

# Numerical Simulations for Magnetohydrodynamics based on Upwind Schemes

Energy vs Entropy

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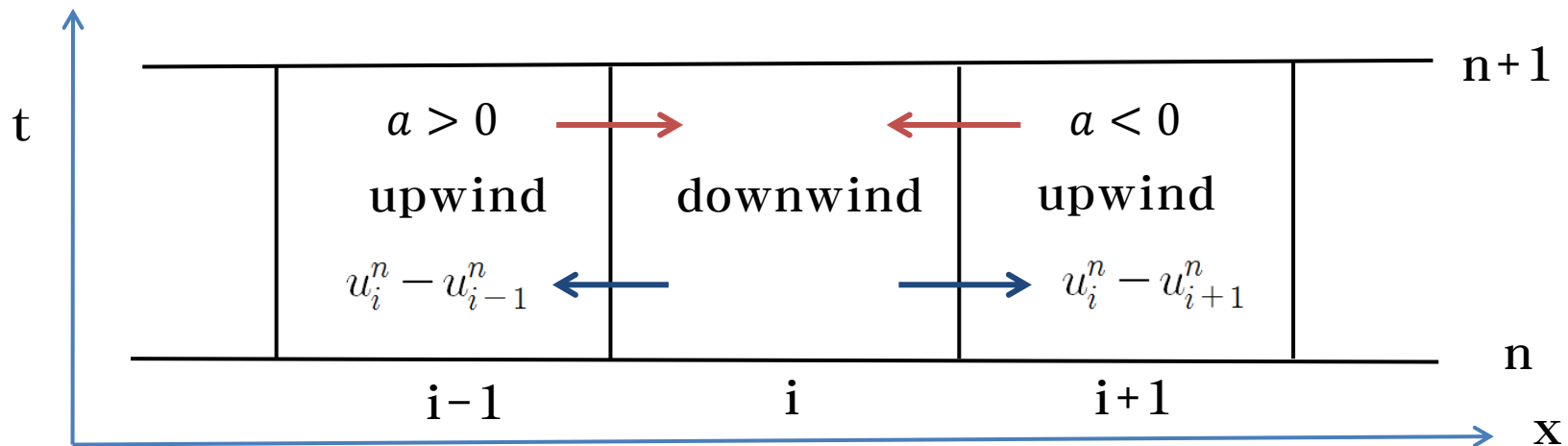
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# Upwind schemes

Numerical methods for solving hyperbolic equations using propagation information

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad \text{one-dimensional linear advection equation}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \quad \text{for } a > 0$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - a \frac{u_i^n - u_{i+1}^n}{\Delta x} = 0 \quad \text{for } a < 0$$



Robust and reliable schemes for gas dynamics

# MHD-E

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = \mathbf{0}$$

state vector  $\vec{q} = \begin{bmatrix} D \\ M_i \\ E \\ B_y \\ B_z \end{bmatrix}$

- Mass
- Momentums
- Total energy
- Magnetic fields

flux vector  $\vec{F} = \begin{bmatrix} Dv_x \\ M_i v_x + p^* \delta_{ij} - B_x B_i \\ E v_x + p^* v_x - B_x (\vec{v} \cdot \vec{B}) \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \end{bmatrix}$

$$D = \rho$$

$$M_i = \rho v_i$$

$$E = \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{2} B^2$$

$\rho$  proper rest mass density  
 $v_i$  fluid velocities

$p^* = p + \frac{1}{2} B^2$  total pressure

# MHD-S

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = 0$$

$$\vec{q} = \begin{bmatrix} D \\ M_i \\ S \\ B_y \\ B_z \end{bmatrix}$$

state vector

- Mass
- Momentums
- Entropy
- Magnetic fields

$$\vec{F} = \begin{bmatrix} Dv_x \\ M_i v_x + p^* \delta_{ij} - B_x B_i \\ Sv_x \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \end{bmatrix}$$

flux vector

$$D = \rho$$

$$M_i = \rho v_i$$

$$S = p \rho^{1-\gamma}$$

$\rho$  proper rest mass density  
 $v_i$  fluid velocities

$$p^* = p + \frac{1}{2} B^2 \quad \text{total pressure}$$

# Jacobian matrix

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = \frac{\partial \vec{q}}{\partial t} + A \frac{\partial \vec{q}}{\partial x} = 0$$

$$A = \frac{\partial \vec{F}}{\partial \vec{q}} = \frac{\partial \vec{F}}{\partial u} \frac{\partial u}{\partial \vec{q}}$$

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} \end{pmatrix}$$

$$\vec{u} = \begin{bmatrix} \rho \\ v_x \\ v_y \\ v_z \\ p \\ B_y \\ B_z \end{bmatrix}$$

parameter vector

# Jacobian matrix

Eigenvalues  $\det(A - a_i I) = 0$

Right Eigenvectors  $(A - a_i) \cdot \vec{R}_i = 0$

Left Eigenvectors  $\vec{L}_i \cdot (A - a_i) = 0$

$$\vec{L} \cdot A \cdot \vec{R} = \Lambda = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{bmatrix}$$

# Eigenvalues

**MHD**

$$c_s^2 = \frac{\gamma p}{\rho} \quad b_i = \frac{B_i}{\sqrt{\rho}}$$

$$a_1 = v_x - \sqrt{\frac{1}{2} [c_s^2 + b^2 + \sqrt{(c_s^2 + b^2)^2 - 4c_s^2 b_x^2}]},$$

-fast magnetosonic mode

$$a_2 = v_x - |b_x|,$$

-Alfven mode

$$a_3 = v_x - \sqrt{\frac{1}{2} [c_s^2 + b^2 - \sqrt{(c_s^2 + b^2)^2 - 4c_s^2 b_x^2}]},$$

-slow magnetosonic mode

$$a_4 = v_x,$$

entropy mode

$$a_5 = v_x + \sqrt{\frac{1}{2} [c_s^2 + b^2 - \sqrt{(c_s^2 + b^2)^2 - 4c_s^2 b_x^2}]},$$

+slow magnetosonic mode

$$a_6 = v_x + |b_x|,$$

+Alfven mode

$$a_7 = v_x + \sqrt{\frac{1}{2} [c_s^2 + b^2 + \sqrt{(c_s^2 + b^2)^2 - 4c_s^2 b_x^2}]}.$$

+fast magnetosonic mode

$$a_1(\overset{-}{fast}) \leq a_2(\overset{-}{Alfven}) \leq a_3(\overset{-}{slow}) \leq a_4(=v_x) \leq a_5(\overset{+}{slow}) \leq a_6(\overset{+}{Alfven}) \leq a_7(\overset{+}{fast})$$

# Eigenvectors

## MHD-E

### 1. compressible mode

$$\vec{R} = \begin{pmatrix} C \\ (\lambda + v_x)C \\ v_y C - Y_y D \\ v_z C - Y_z D \\ Y_y / \sqrt{\rho} \\ Y_z / \sqrt{\rho} \\ (\lambda^2 + \lambda v_x + \frac{1}{2}v^2 - \frac{\gamma-2}{\gamma-1}c_s^2)C - (Y_y v_y + Y_z v_z)D \end{pmatrix}$$

$$\vec{L} = \frac{1}{2c_s^2} \begin{pmatrix} [\frac{1}{2}(\gamma-1)v^2 - \lambda v_x]C + (Y_y v_y + Y_z v_z)D \\ [\lambda - (\gamma-1)v_x]C \\ (1-\gamma)v_y C - Y_y D \\ (1-\gamma)v_z C - Y_z D \\ B_y(1-\gamma)C + Y_y \sqrt{\rho} \\ B_z(1-\gamma)C + Y_z \sqrt{\rho} \\ (\gamma-1)C \end{pmatrix}^T$$

### 2. Alfven mode

$$\vec{R} = [0, 0, -\beta_z, \beta_y, \beta_z D / \sqrt{\rho}, -\beta_z D / \sqrt{\rho}, \beta_y v_z - \beta_z v_y]^T$$

$$\vec{L} = \frac{1}{2} [\beta_z v_y - \beta_y v_z, 0, -\beta_z, \beta_y, \beta_z D \sqrt{\rho}, -\beta_y D \sqrt{\rho}, 0]$$

### 3. Entropy mode

$$\vec{R} = [1, v_x, v_y, v_z, 0, 0, \frac{1}{2}v^2]^T, \quad \vec{L} = \frac{(\gamma-1)}{c_s^2} \left[ \frac{c_s^2}{(\gamma-1)} - \frac{1}{2}v^2, v_x, v_y, v_z, B_y, B_z, -1 \right]$$

$$\beta_y = \frac{B_y}{\sqrt{B_y^2 + B_z^2}}, \quad \beta_z = \frac{B_z}{\sqrt{B_y^2 + B_z^2}}, \quad \alpha_s = \frac{\sqrt{\lambda_f^2 - c_s^2}}{\sqrt{\lambda_f^2 - \lambda_s^2}}, \quad \alpha_f = \frac{\sqrt{c_s^2 - \lambda_s^2}}{\sqrt{\lambda_f^2 - \lambda_s^2}}$$

for fast mode

$$C = \alpha_f, \quad Y_y = c_s \alpha_s \beta_y, \quad Y_z = c_s \alpha_s \beta_z, \quad D = \frac{b_x}{\lambda_f} = \pm \frac{\lambda_s}{c_s} \text{sgn}(B_x)$$

for slow mode

$$C = \alpha_s, \quad Y_y = -c_s \alpha_f \beta_y, \quad Y_z = -c_s \alpha_f \beta_z, \quad D = \frac{b_x}{\lambda_s} = \pm \frac{\lambda_f}{c_s} \text{sgn}(B_x)$$

for Alfven mode

$$D = \pm \text{sgn}(B_x)$$



# Eigenvectors

$$\beta_y = \frac{B_y}{\sqrt{B_y^2 + B_z^2}}, \quad \beta_z = \frac{B_z}{\sqrt{B_y^2 + B_z^2}}, \quad \alpha_s = \frac{\sqrt{\lambda_f^2 - c_s^2}}{\sqrt{\lambda_f^2 - \lambda_s^2}}, \quad \alpha_f = \frac{\sqrt{c_s^2 - \lambda_s^2}}{\sqrt{\lambda_f^2 - \lambda_s^2}}$$

for fast mode

$$C = \alpha_f, \quad Y_y = c_s \alpha_s \beta_y, \quad Y_z = c_s \alpha_s \beta_z, \quad D = \frac{b_x}{\lambda_f} = \pm \frac{\lambda_s}{c_s} \operatorname{sgn}(B_x)$$

for slow mode

$$C = \alpha_s, \quad Y_y = -c_s \alpha_f \beta_y, \quad Y_z = -c_s \alpha_f \beta_z, \quad D = \frac{b_x}{\lambda_s} = \pm \frac{\lambda_f}{c_s} \operatorname{sgn}(B_x)$$

for Alfvén mode

$$D = \pm \operatorname{sgn}(B_x)$$

## MHD-S

### 1. compressible mode

$$\vec{R} = \begin{pmatrix} C \\ (\lambda + v_x)C \\ v_y C - Y_y D \\ v_z C - Y_z D \\ Y_y / \sqrt{\rho} \\ Y_z / \sqrt{\rho} \\ c_s^2 C / \gamma \end{pmatrix}, \quad \vec{L} = \frac{1}{2c_s^2} \begin{pmatrix} [c_s^2(\gamma - 1)/\gamma - \lambda v_x]C + (Y_y v_y + Y_z v_z)D \\ \lambda C \\ -Y_y D \\ -Y_z D \\ Y_y \sqrt{\rho} \\ Y_z \sqrt{\rho} \\ C \end{pmatrix}^T$$

### 2. Alfvén mode

$$\vec{R} = [0, 0, -\beta_z, \beta_y, \beta_z D / \sqrt{\rho}, -\beta_z D / \sqrt{\rho}, 0]^T$$

$$\vec{L} = \frac{1}{2} [\beta_z v_y - \beta_y v_z, 0, -\beta_z, \beta_y, \beta_z D \sqrt{\rho}, -\beta_y D \sqrt{\rho}, 0]$$

### 3. Entropy mode

$$\vec{R} = [1, v_x, v_y, v_z, 0, 0, c_s^2(1 - \gamma)/\gamma]^T, \quad \vec{L} = [1/\gamma, 0, 0, 0, 0, 0, -1/c_s^2]$$

# Numerical Simulations

Total Variation Diminishing (TVD)

Harten 1983 (HD), Ryu et al 1995 (MHD)

Weighted Essentially Non-Oscillatory (WENO)

Jiang & Shu 1996 (HD), Jiang & Wu 1999 (MHD)

TVD (Space2-Time2)

WENO3-RK3 (Space3-Time3)

WENO5-RK4 (Space5-Time4)

# Numerical Simulations

MHD-E

MHD-S

TVD\_E

TVD\_S

WENO3-RK3\_E

WENO3-RK3\_S

WENO5-RK4\_E

WENO5-RK4\_S

# MHD Alfven wave tests

Initially

Periodic boundary

$$\rho = \rho_0 + \delta\rho$$

$$\rho_0 = 1, \delta\rho = 0$$

$$k = \frac{2\pi}{L}$$

$$v_x = v_{x0} + \delta v_x$$

$$v_{x0} = 0, \delta v_x = 0$$

$$\gamma = 5/3$$

$$v_y = v_{y0} + \delta v_y$$

$$v_{y0} = 0, \delta v_y = 0$$

$$c_{s0} = \sqrt{\gamma p_0 / \rho_0} = 1$$

$$v_z = v_{z0} + \delta v_z$$

$$v_{z0} = 0, \delta v_z = 10^{-4} \cos(kx)$$

$$p = p_0 + \delta p$$

$$p_0 = 1/\gamma, \delta p = 0$$

$$\frac{B_{x0}}{\sqrt{\rho_0}} = 1 \quad \text{Alfven speed}$$

$$B_x = B_{x0}$$

$$B_{x0} = 1$$

$$t = 100 \quad 100 \text{ times}$$

$$B_y = B_{y0} + \delta B_y$$

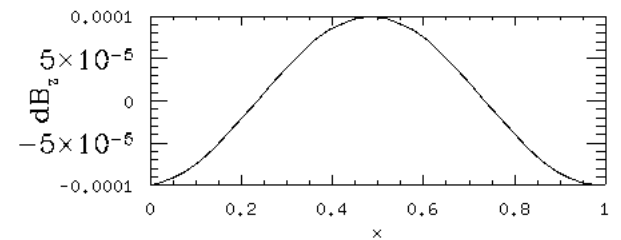
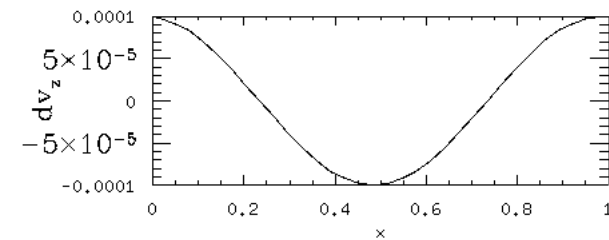
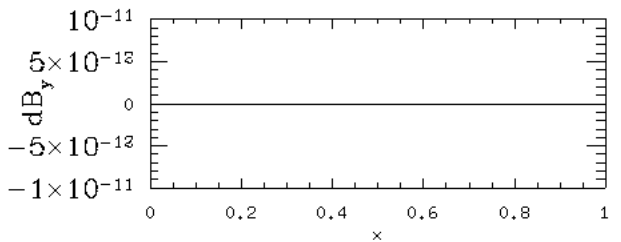
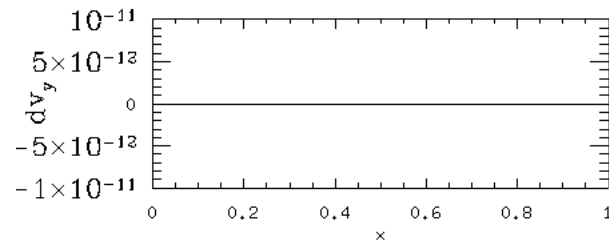
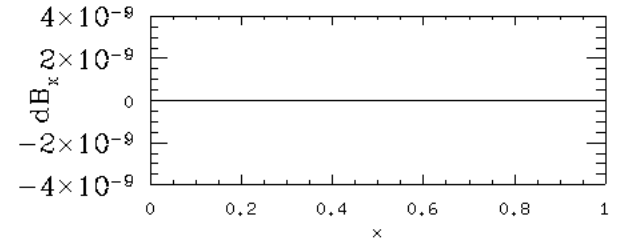
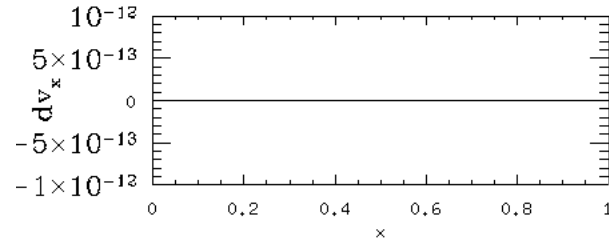
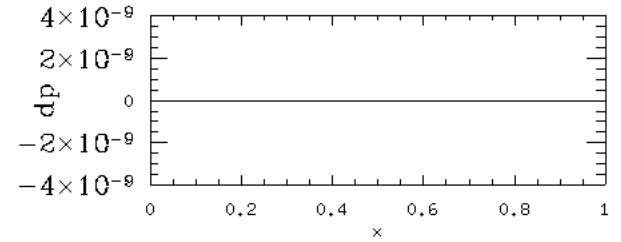
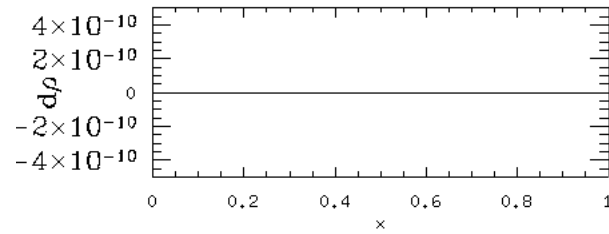
$$B_{y0} = 0.5, \delta B_y = 0$$

$$B_z = B_{z0} + \delta B_z$$

$$B_{z0} = 0, \delta B_z = -\delta v_z \sqrt{\rho_0} + \text{Alfven wave}$$

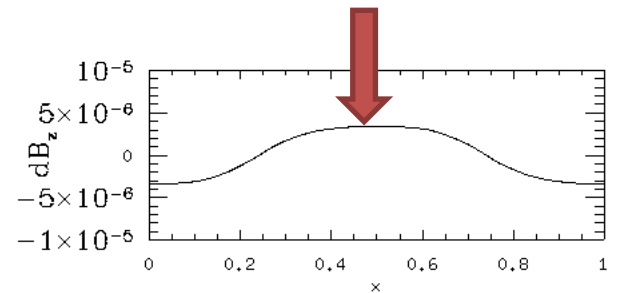
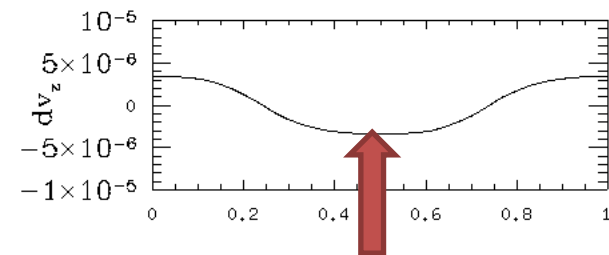
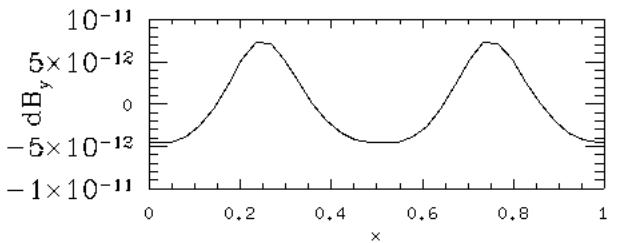
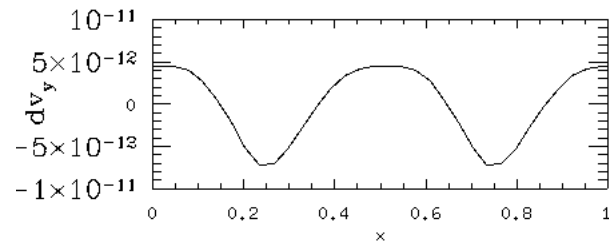
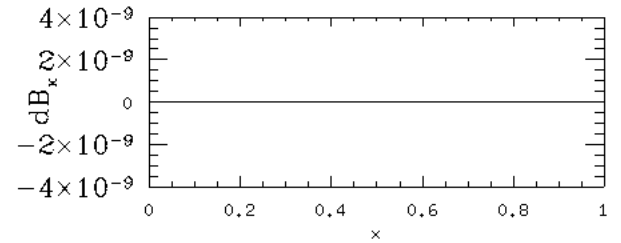
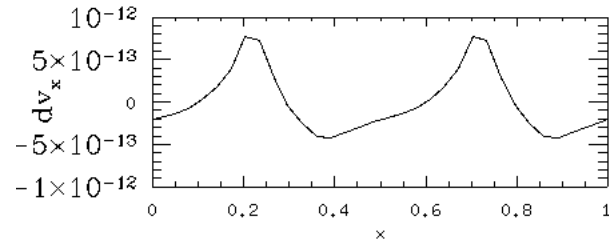
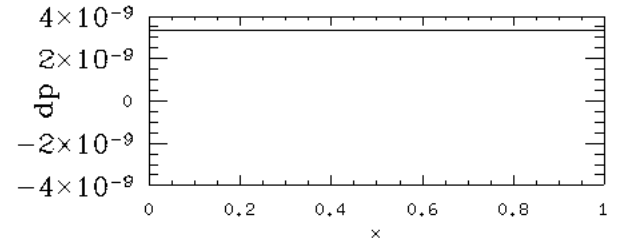
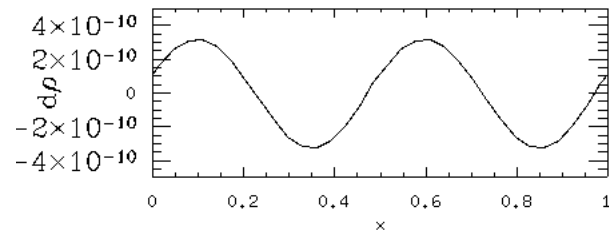
# MHD Alfven wave tests

Initially



# MHD Alfven wave tests

## TVD\_E



# MHD Alfven wave tests

Initially

Periodic boundary

$$\rho = \rho_0 + \delta\rho$$

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$$v_y = v_{y0} + \delta v_y$$

$$v_{y0} = 0, \delta v_y = 0$$

$$c_{s0} = \sqrt{\gamma p_0 / \rho_0} = 1$$

$$v_z = v_{z0} + \delta v_z$$

$$v_{z0} = 0, \delta v_z = 10^{-4} \cos(kx)$$

$$p = p_0 + \delta p$$

$$p_0 = 1/\gamma, \delta p = 0$$

$$\frac{B_{x0}}{\sqrt{\rho_0}} = 1 \quad \text{Alfven speed}$$

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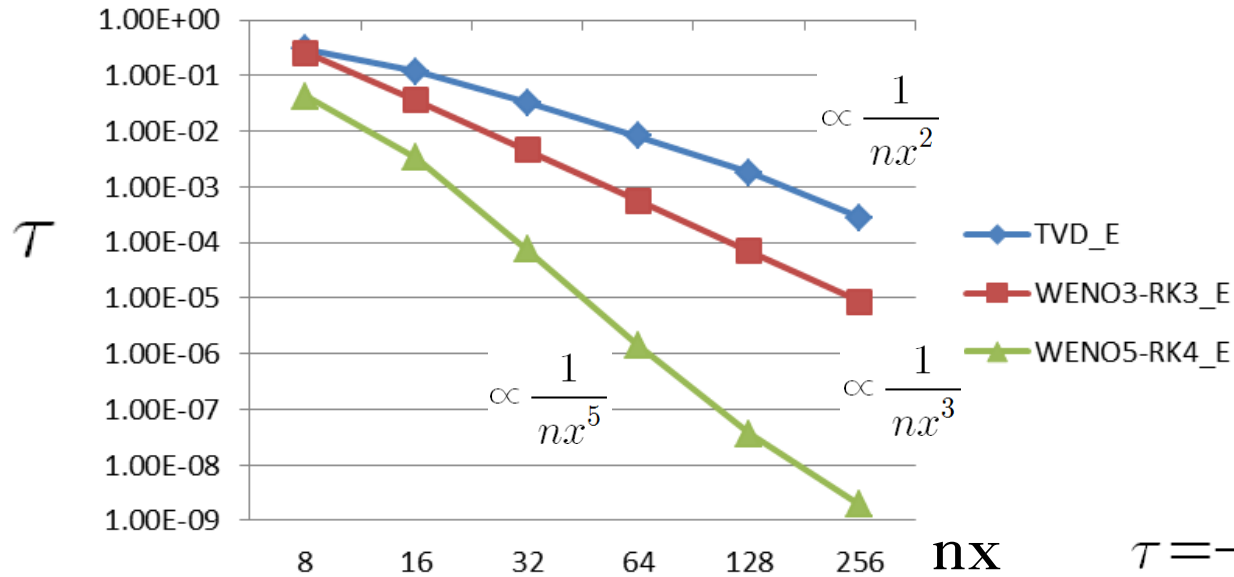
$$B_z = B_{z0} + \delta B_z$$

$$B_{z0} = 0, \delta B_z = -\delta v_z \sqrt{\rho_0} + \text{Alfven wave}$$

Numerical dissipation

$$\frac{v_{z,rms}(t)}{v_{z,rms}(t=0)} = e^{-t\tau} \quad \tau = -\frac{1}{t} \ln \left( \frac{v_{z,rms}(t)}{v_{z,rms}(t=0)} \right) \quad \text{Decay rate}$$

# MHD Alfven wave tests



**MHD-E**

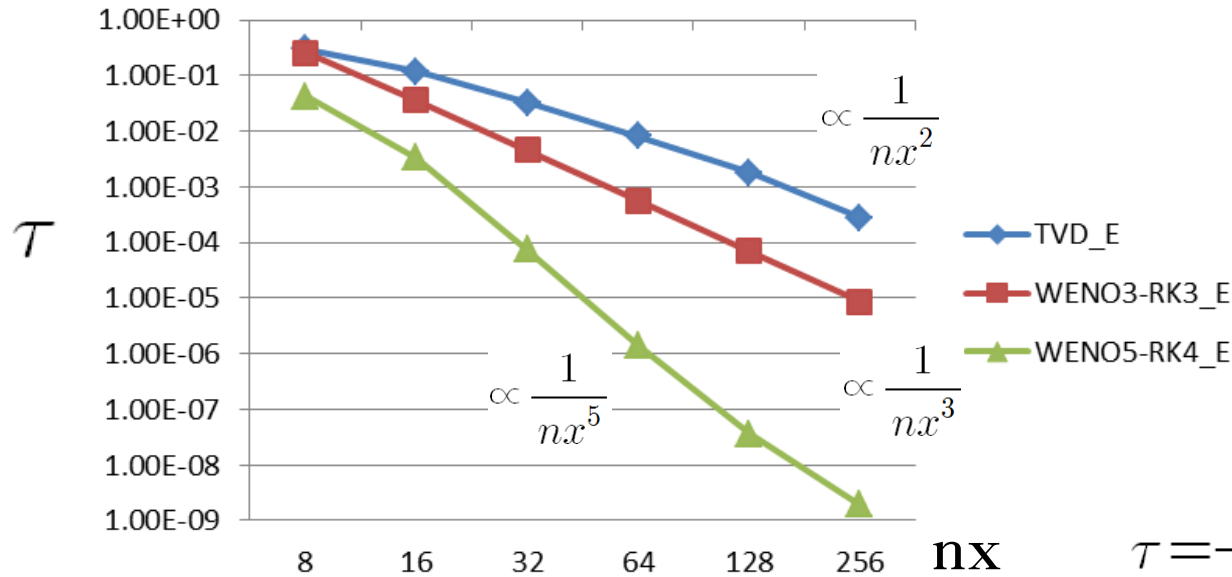
$$\tau = -\frac{1}{t} \ln \left( \frac{v_{z,rms}(t)}{v_{z,rms}(t=0)} \right)$$

Decay rate

$nx$  : resolution



# MHD Alfven wave tests

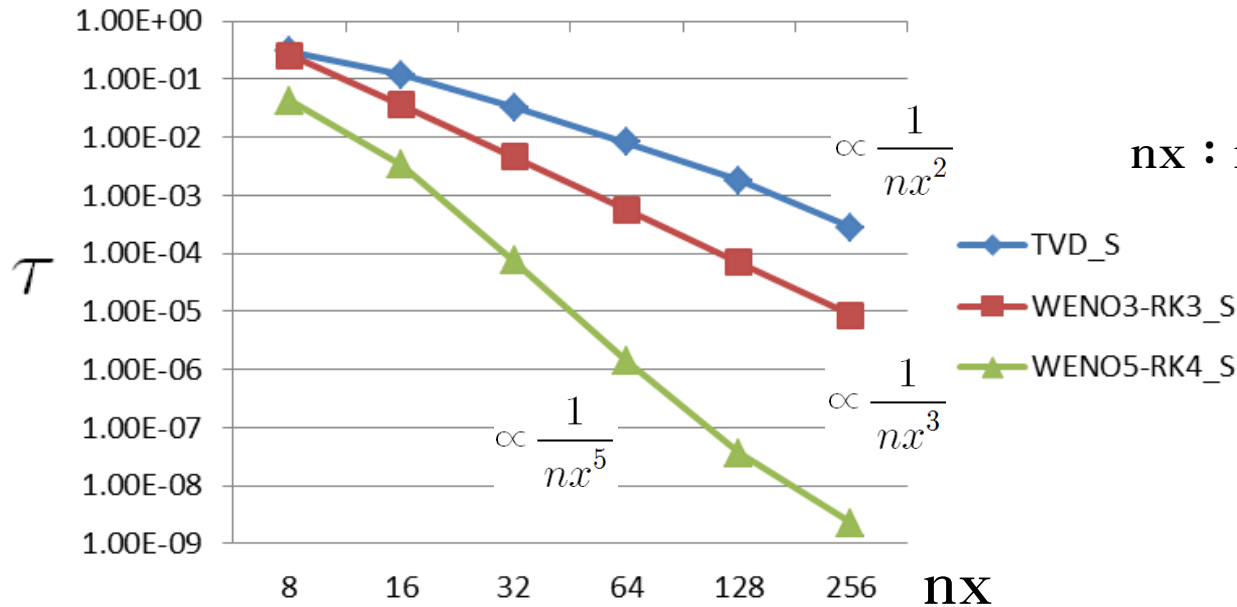


MHD-E

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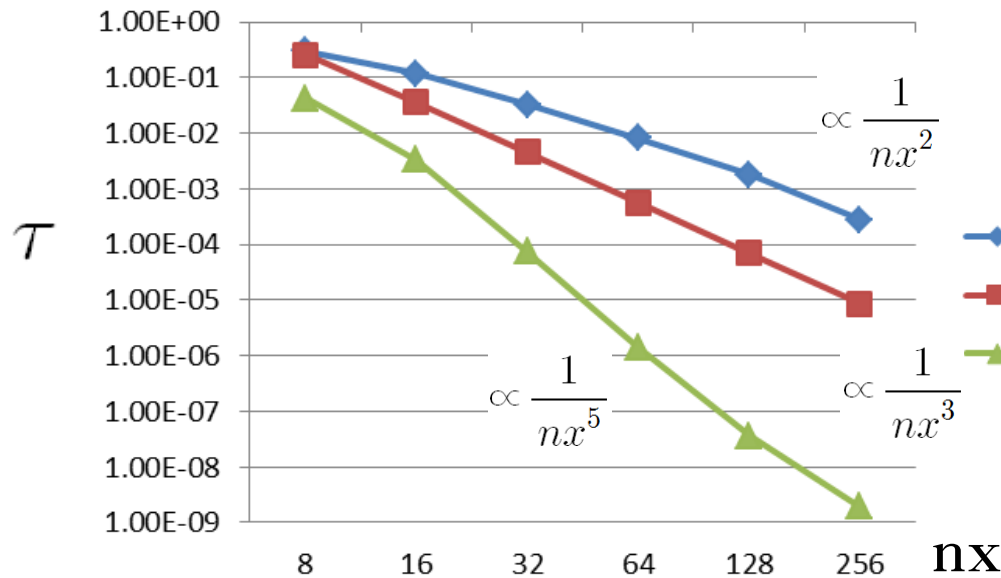
Decay rate

nx : resolution



MHD-S

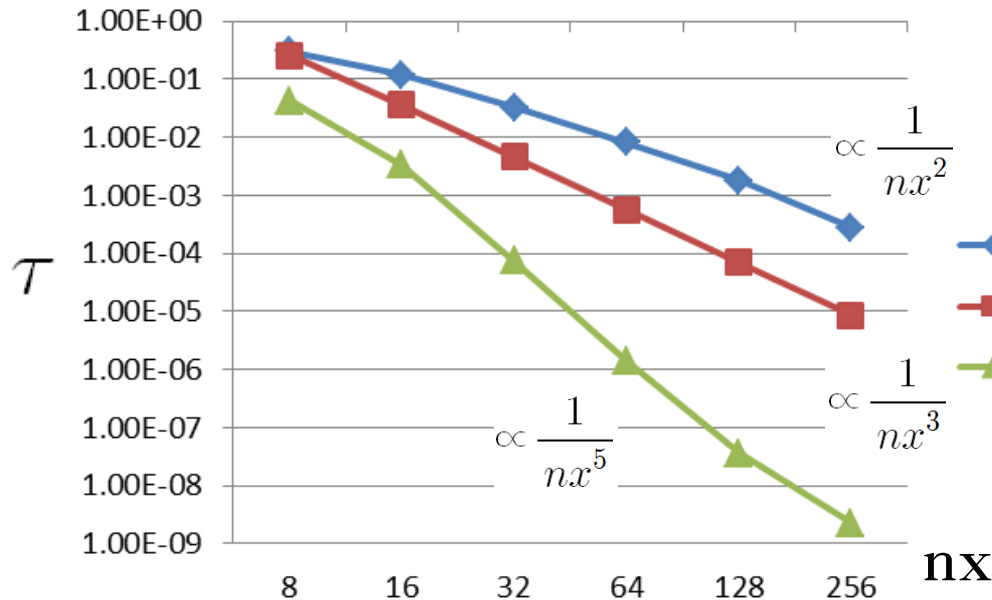
# MHD Alfven wave tests



$$\tau = -\frac{1}{t} \ln \left( \frac{v_{z,rms}(t)}{v_{z,rms}(t=0)} \right)$$

Decay rate

nx : resolution



**High order scheme  
is better!**

# MHD Alfven wave tests

Initially

Periodic boundary

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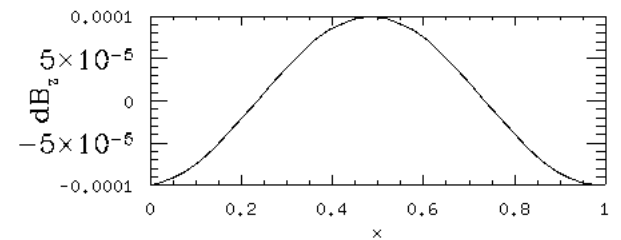
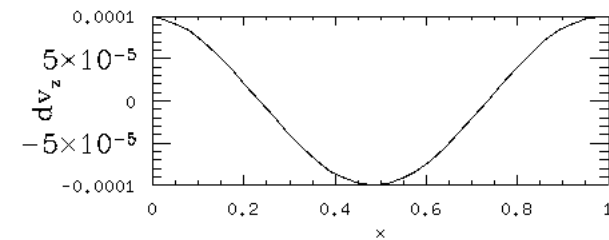
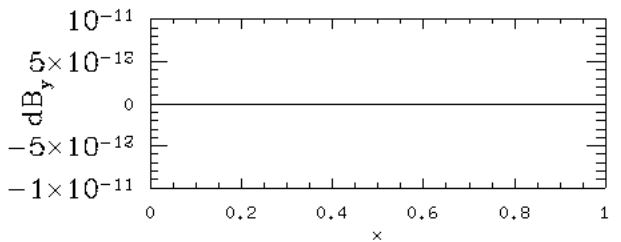
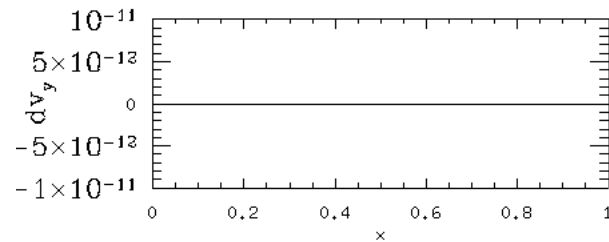
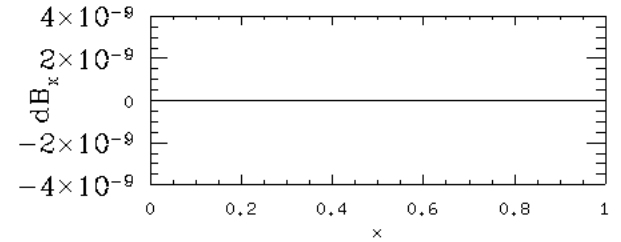
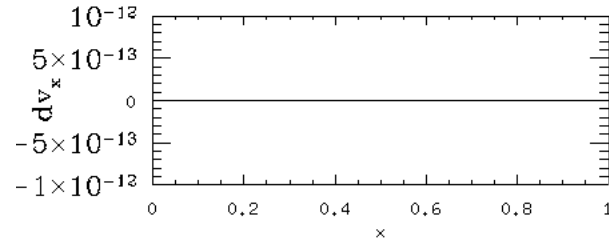
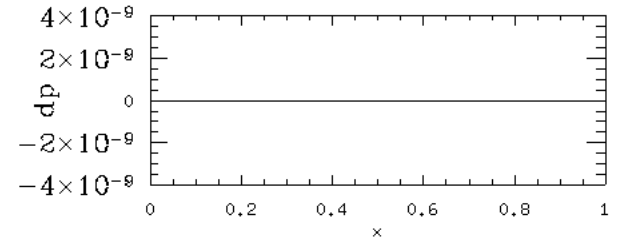
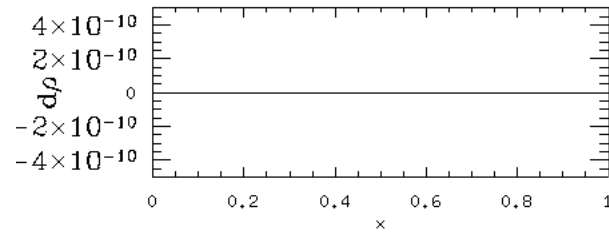
$$B_{z0} = 0, \delta B_z = -\delta v_z \sqrt{\rho_0} + \text{Alfven wave}$$

$$E = \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{2} B^2 \longrightarrow p = (\gamma - 1) \left[ E - \frac{1}{2} (\rho v^2 + B^2) \right]$$

$$S = p \rho^{1-\gamma} \longrightarrow p = S \rho^{\gamma-1}$$

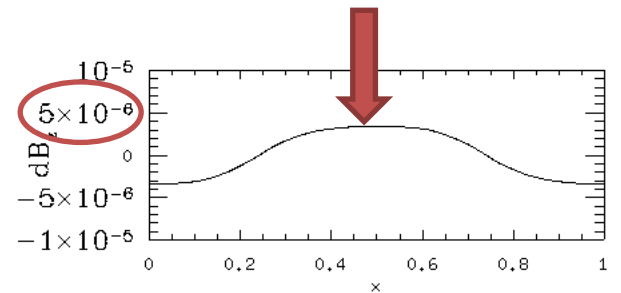
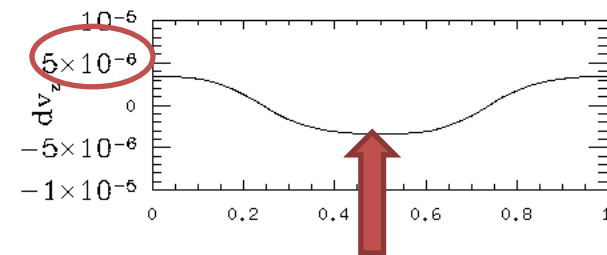
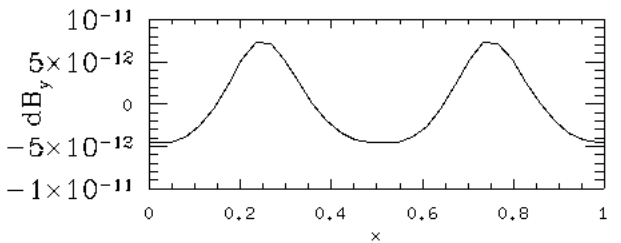
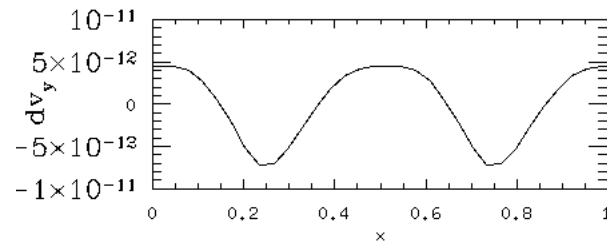
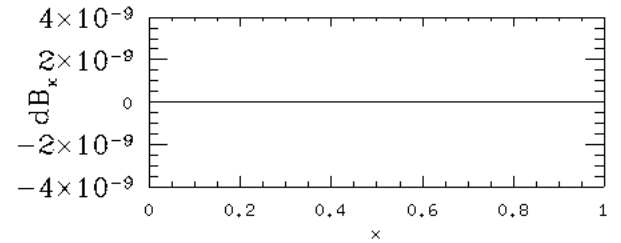
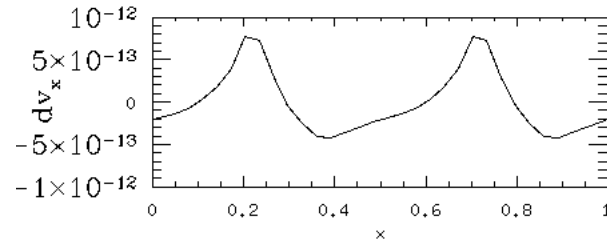
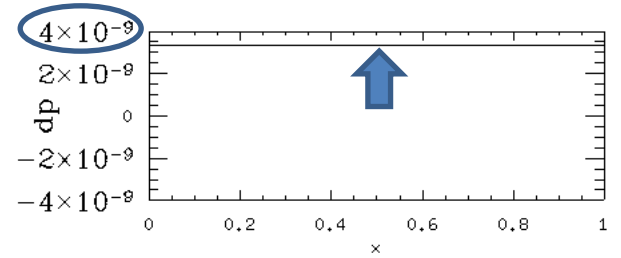
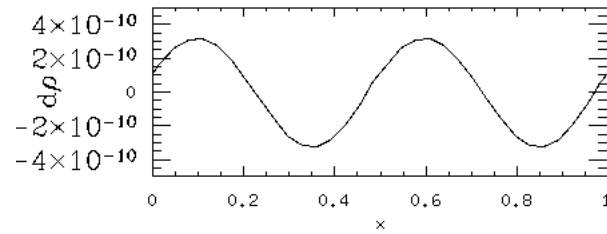
# MHD Alfven wave tests

## Input



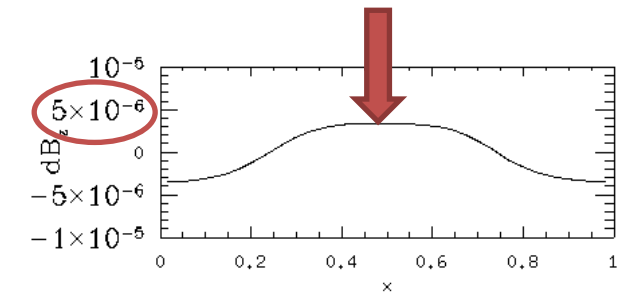
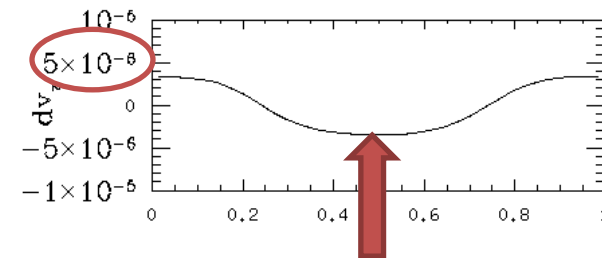
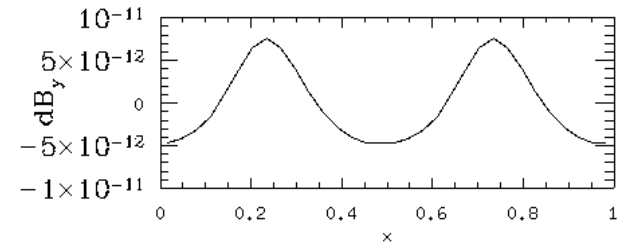
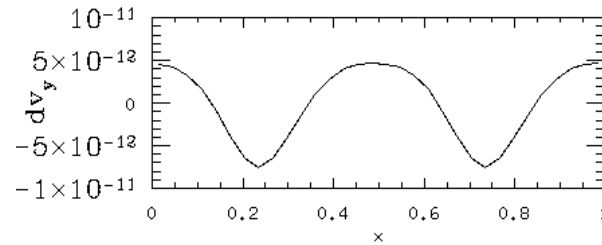
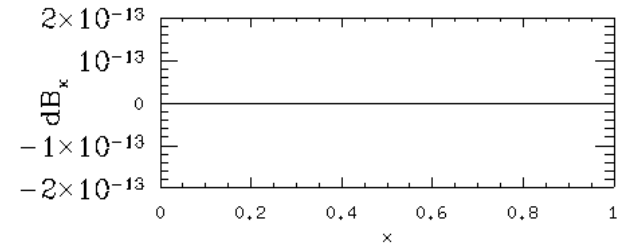
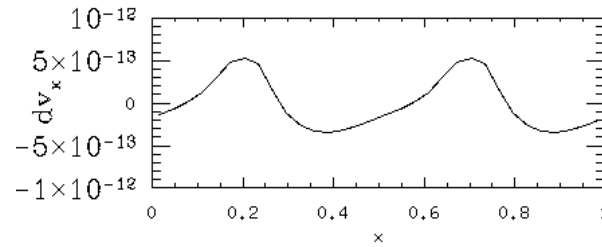
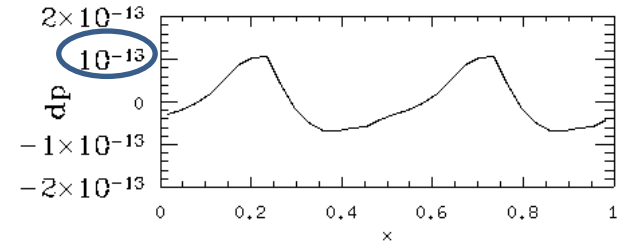
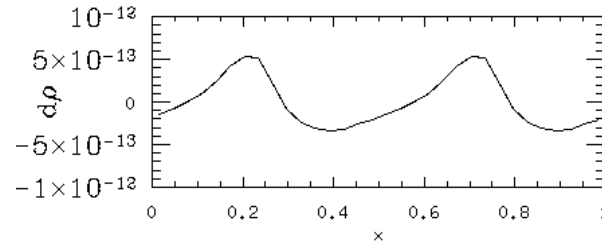
# MHD Alfven wave tests

## TVD\_E



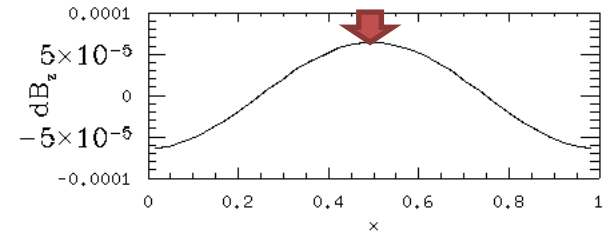
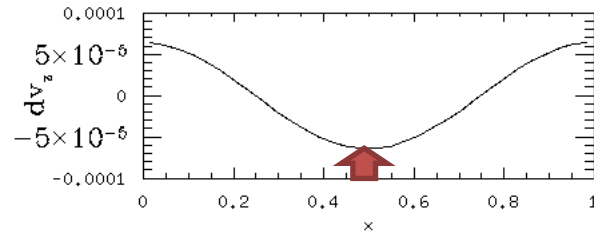
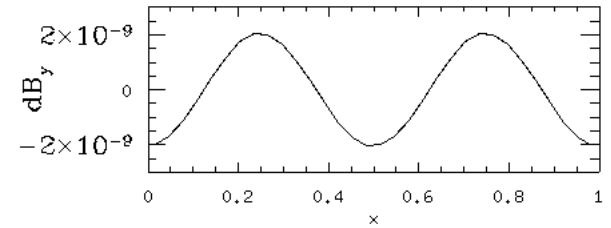
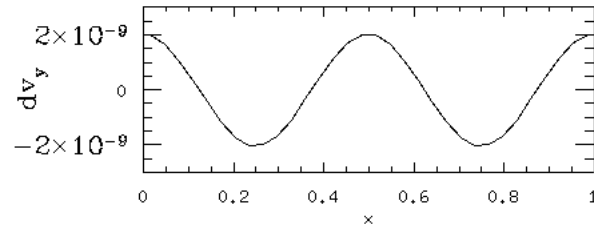
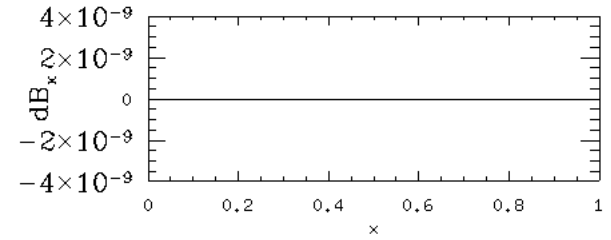
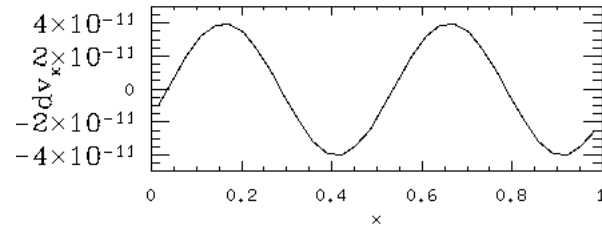
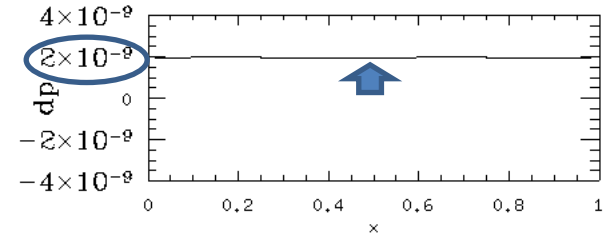
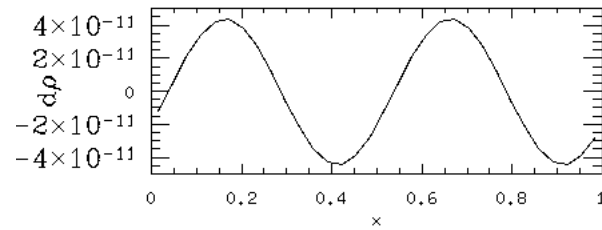
# MHD Alfven wave tests

## TVD\_S



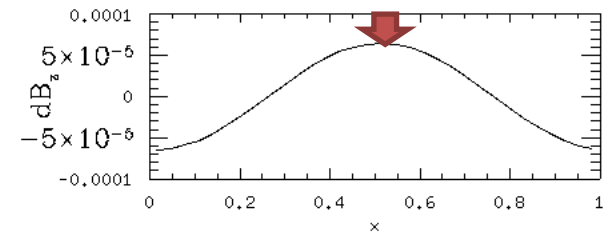
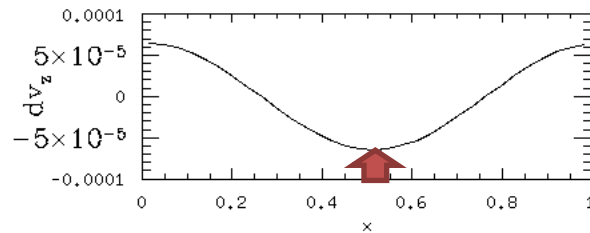
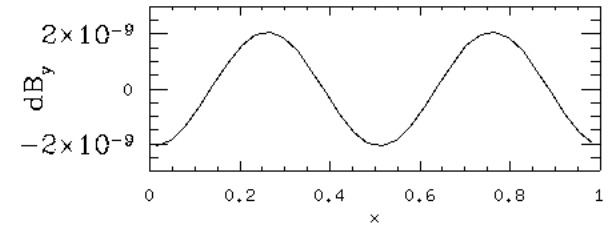
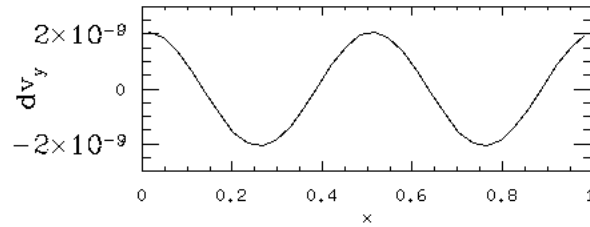
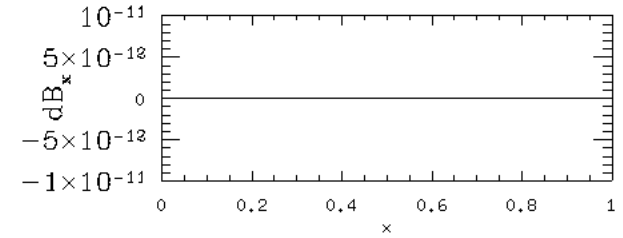
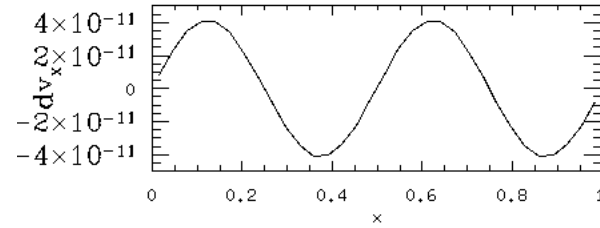
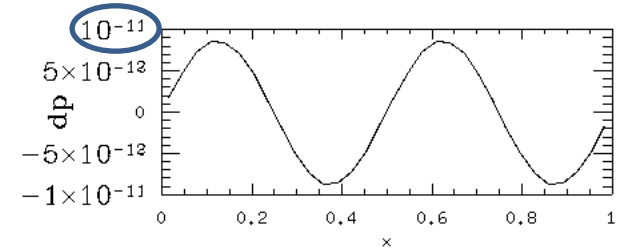
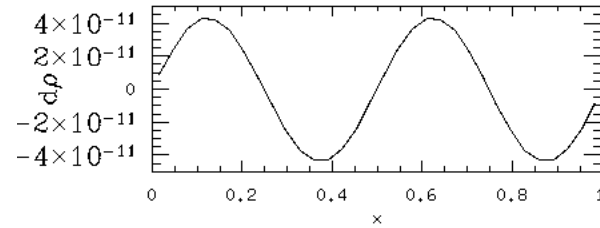
# MHD Alfvén wave tests

WENO3  
-RK3\_E



# MHD Alfven wave tests

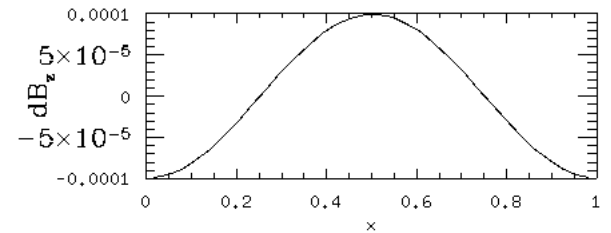
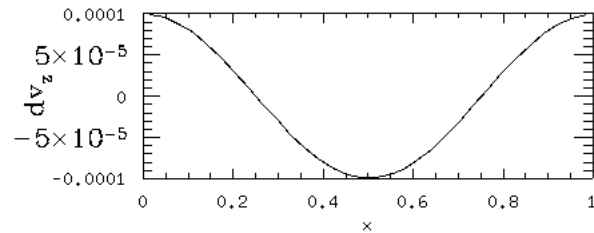
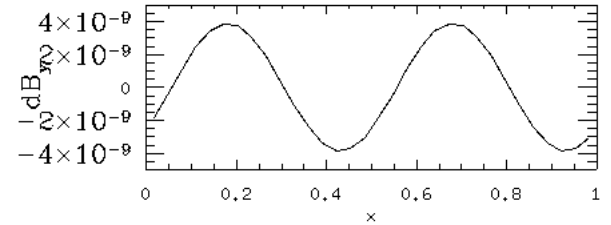
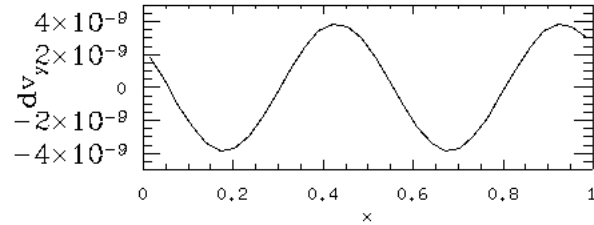
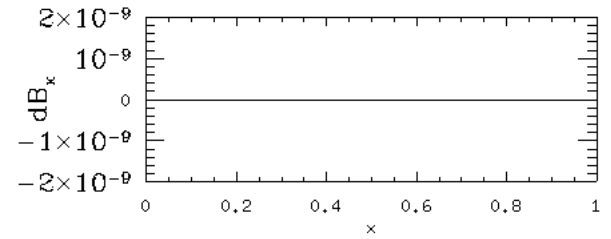
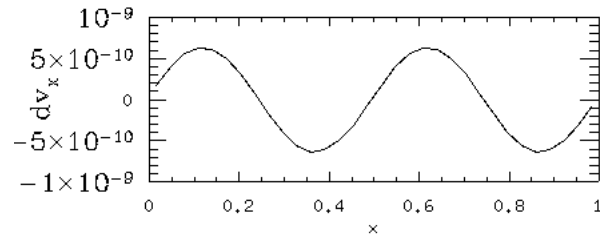
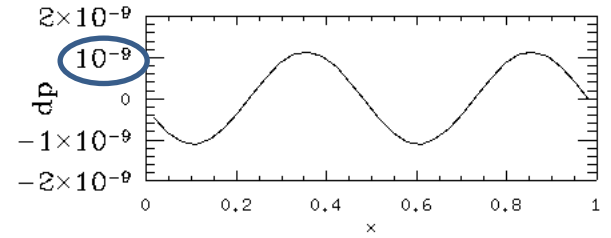
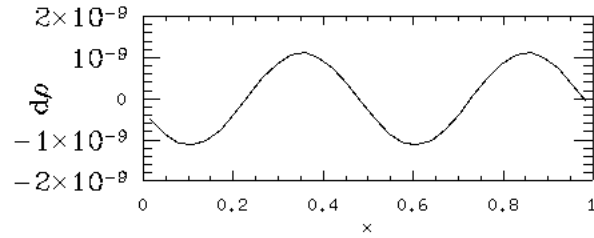
WENO3  
-RK3\_S





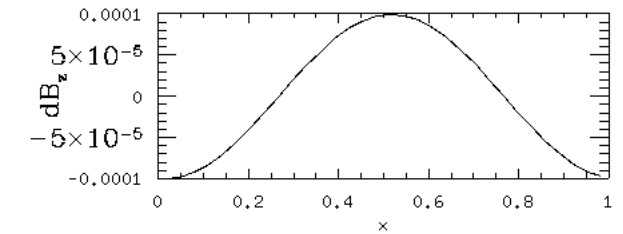
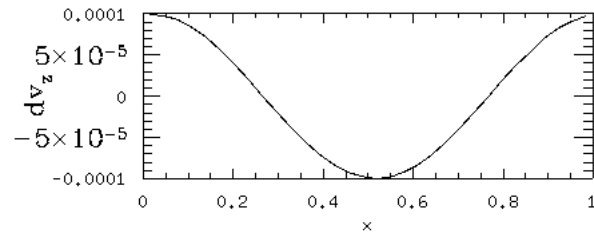
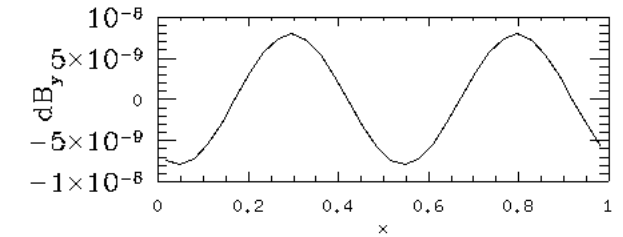
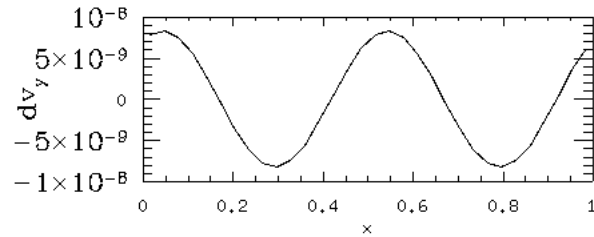
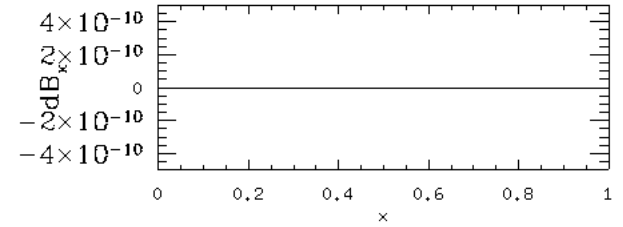
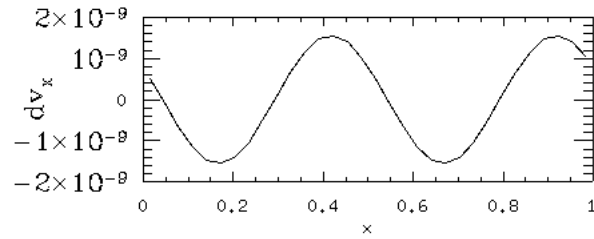
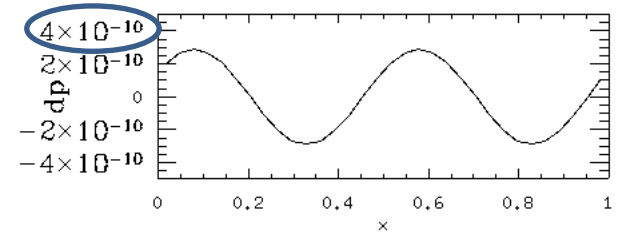
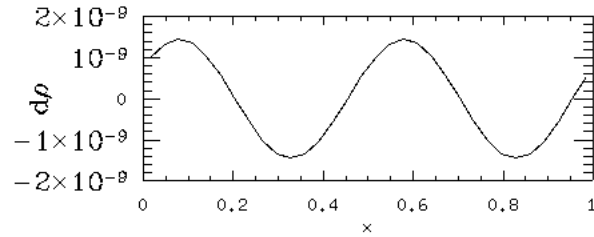
# MHD Alfven wave tests

WENO5  
-RK4\_E



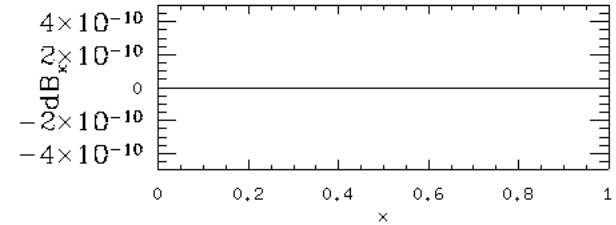
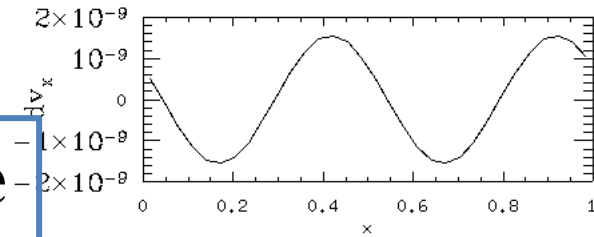
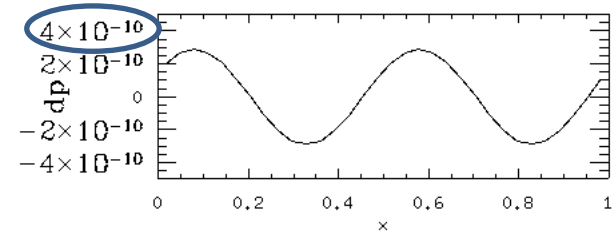
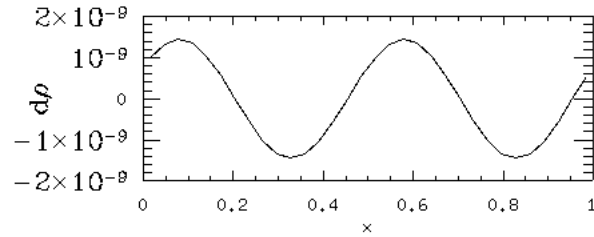
# MHD Alfven wave tests

WENO5  
-RK4\_S

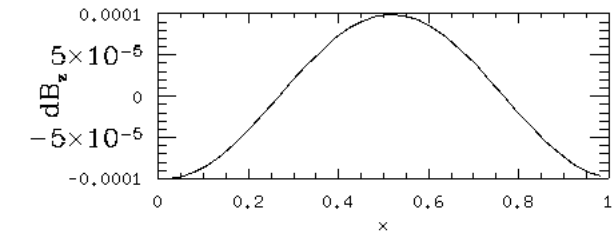
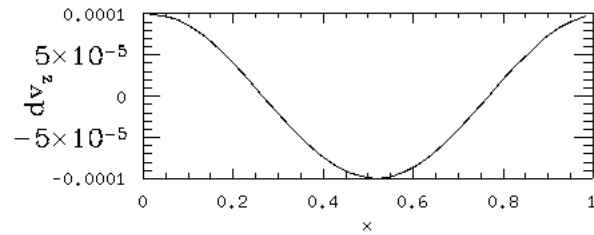
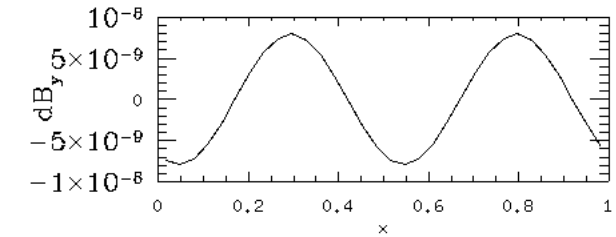
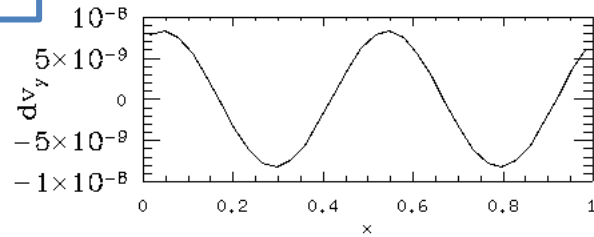


# MHD Alfven wave tests

WENO5  
-RK4\_S



MHD-S code  
is better!



# Conclusion

**MHD-E**

Shocks

$$p \sim E$$

low  $\beta$  ( $p_g \sim p_m$ )

**MHD-S**

No shocks

$$p \ll E$$

high  $\beta$  ( $p_g \ll p_m$ )

In the shock regimes  MHD-E

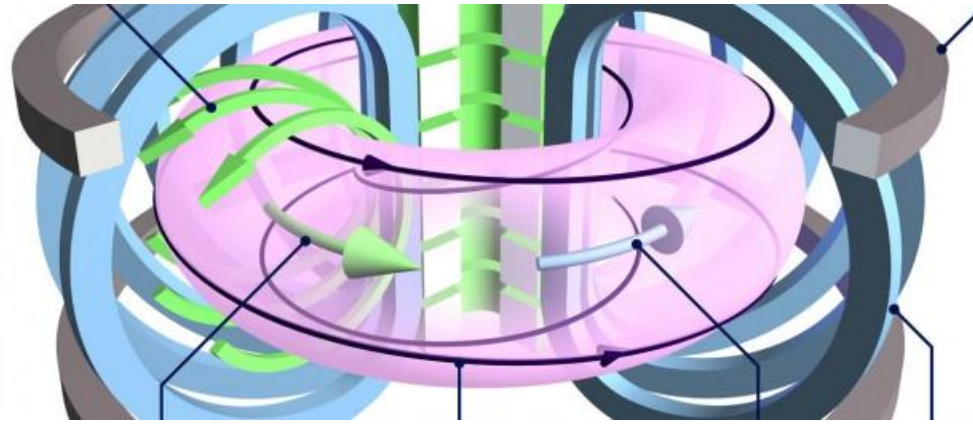
Else where  MHD-S

using TVD, WENO3, WENO5

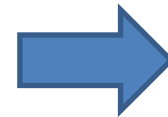
# Conclusion

Application to Nuclear Fusion

: in tokamak



No shocks,  $p \ll E$ , high  $\beta$  ( $p_g \ll p_m$ )



MHD-S

Propagate waves many times  
→ small numerical dissipation



WENO

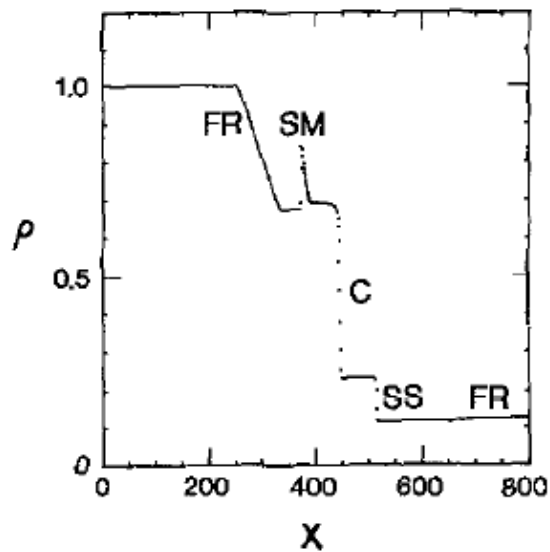
**Develop Korea tokamak simulations!**

**Thank you :) )**

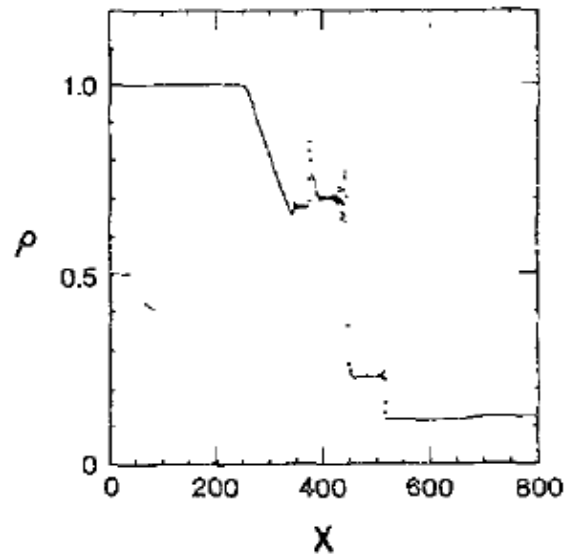
# Upwind schemes

(Brio & Wu 1988)

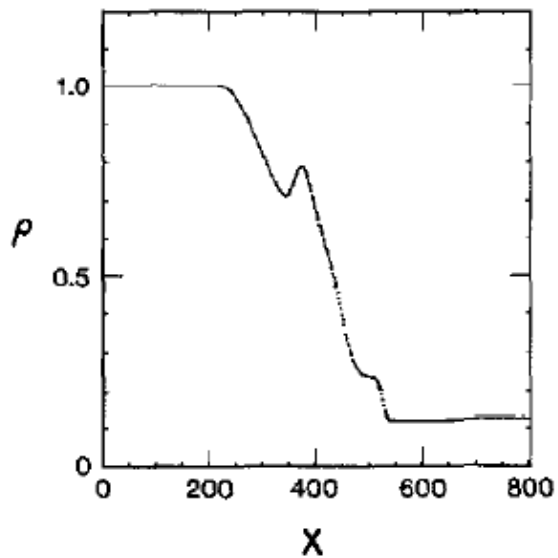
Second-order upwind scheme



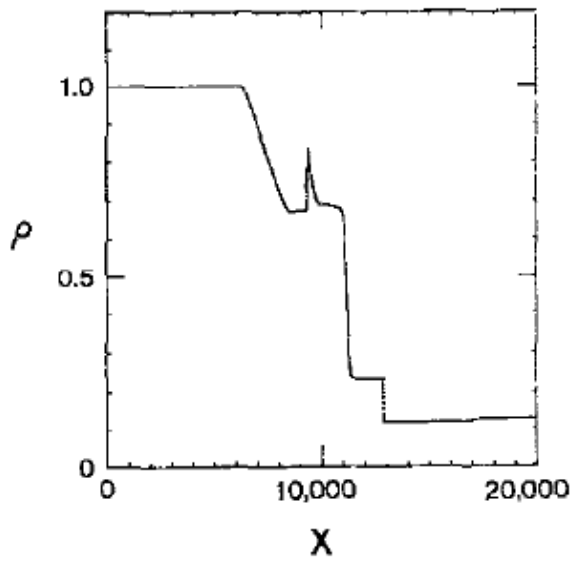
Lax-Wendroff scheme



Lax-Friedrichs scheme



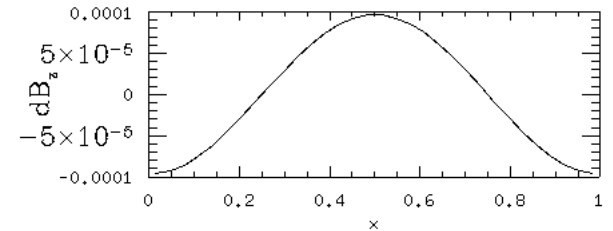
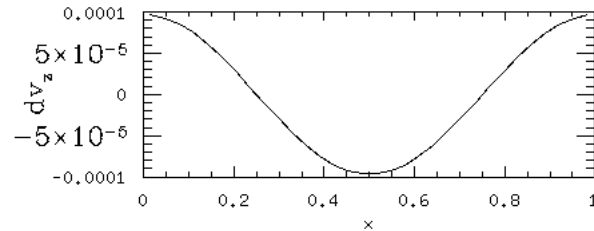
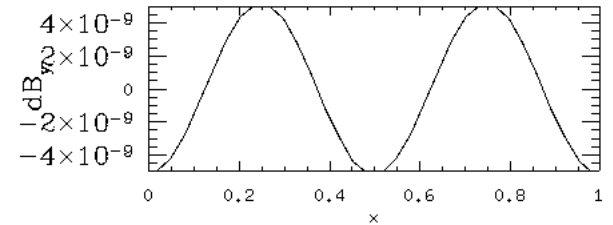
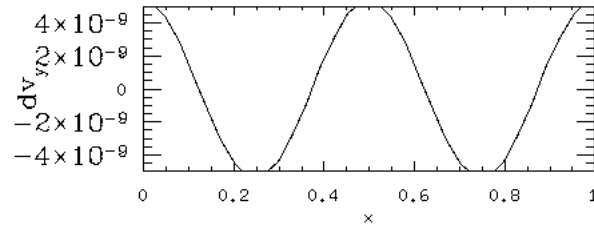
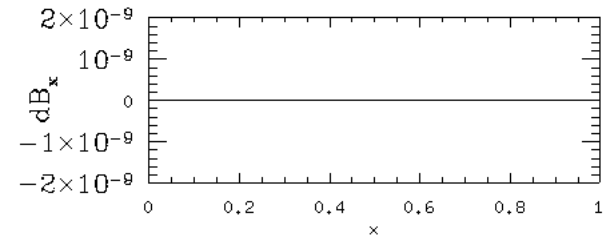
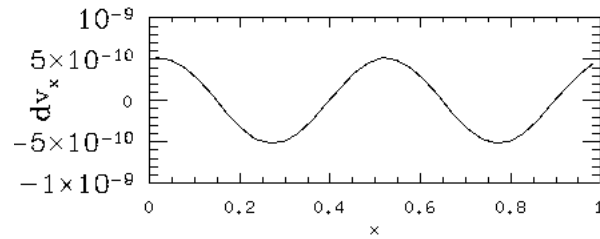
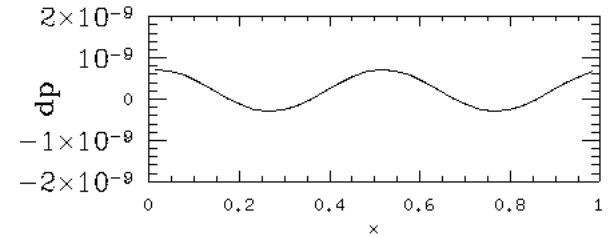
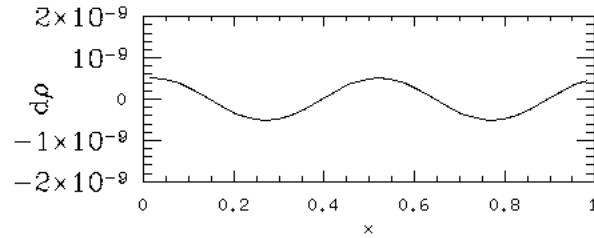
Lax-Friedrichs scheme with 20,000 points



# MHD Alfven wave tests

WENO5  
-RK4\_E

1000 times





# MHD Alfven wave tests

WENO5  
-RK4\_S

1000 times

