

Numerical relativistic hydrodynamics with ADER-WENO adaptive mesh refinement

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with: M. Dumbser

Outline

- 1 Motivations
- 2 The numerical method (ADER approach)
 - Finite volume scheme
 - Local space-time Discontinuous Galerkin predictor
- 3 Adaptive Mesh Refinement
 - AMR implementation
 - Local Timestepping
- 4 The Richtmyer–Meshkov instability

Motivations

Many physical problems show great disparities in the spatial and temporal scales (**multiscale problems**), which a static grid approach cannot treat efficiently.

- Turbulence
- Magnetic instabilities
- Magnetic reconnection
- Binary mergers
- Circumbinary discs
- Relativistic jets
- ...

Adaptive Mesh Refinement (AMR), namely the possibility to change the computational grid dynamically in space and in time, becomes necessary.

High Order Methods can also help substantially when very small details need to be solved.

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- AMR+High Order Methods (relativistic case)
 - FLASH code by Fryxell et al. (2000), ApJ Suppl., 131, 273
 - Zhang & MacFadyen (2006), ApJ Suppl., 164, 255
 - RENZO code by Wang et al. (2008), ApJ, 176, 467
 - PLUTO code by Mignone et al. (2012), ApJ Suppl., 198, 7

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- A particularly appealing implementation of High order methods is represented by **ADER schemes**, which are one-step time-update schemes.
 - Original version: use the Lax-Wendroff procedure [Toro et al. (2001); Titarev and Toro (2002) ...]
 - Modern version: use a weak integral formulation of the governing PDE [Dumbser et al. (2008) JCP, 227, 3971; Balsara et al (2009) JCP, 228, 2480]

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- **AMR+ADER schemes**.... presented very recently for the non-relativistic Euler equations by
 - Dumbser, Zanotti, Hidalgo, Balsara (2013), JCP, 248, 257-286
 - and extended to the relativistic regime at this conference for the first time

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Finite volume scheme

We consider **hyperbolic systems** of balance laws in Cartesian coordinates

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} + \frac{\partial \mathbf{h}}{\partial z} = 0$$

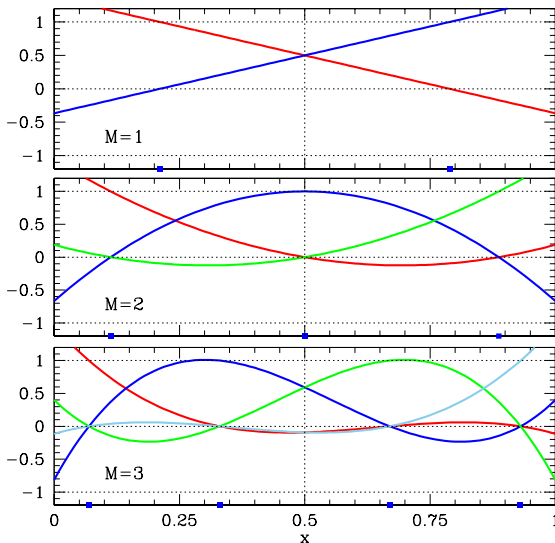
We use a **Standard Finite Volume discretization**

$$\begin{aligned} \mathbf{u}_{ijk}^{n+1} &= \mathbf{u}_{ijk}^n - \frac{\Delta t}{\Delta x_i} \left(\mathbf{f}_{i+\frac{1}{2},jk} - \mathbf{f}_{i-\frac{1}{2},jk} \right) - \frac{\Delta t}{\Delta y_j} \left(\mathbf{g}_{i,j+\frac{1}{2},k} - \mathbf{g}_{i,j-\frac{1}{2},k} \right) \\ &\quad - \frac{\Delta t}{\Delta z_k} \left(\mathbf{h}_{ij,k+\frac{1}{2}} - \mathbf{h}_{ij,k-\frac{1}{2}} \right) + \Delta t \mathbf{S}_{ijk}, \end{aligned}$$

over the control volumes $l_{ijk} = [x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}; y_{j+\frac{1}{2}}] \times [z_{k-\frac{1}{2}}; z_{k+\frac{1}{2}}]$, with

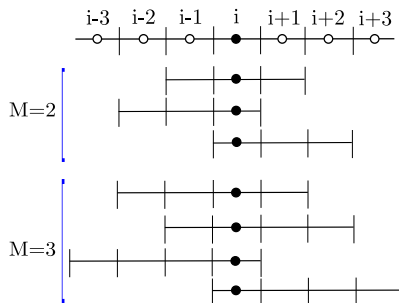
$$\begin{aligned} \mathbf{u}_{ijk}^n &= \frac{1}{\Delta x_i} \frac{1}{\Delta y_j} \frac{1}{\Delta z_k} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \mathbf{u}(x, y, z, t^n) dz dy dx \\ \mathbf{f}_{i+\frac{1}{2},jk} &= \frac{1}{\Delta t} \frac{1}{\Delta y_j} \frac{1}{\Delta z_k} \int_{t^n}^{t^{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \tilde{\mathbf{f}} \left(\mathbf{q}_h^-(x_{i+\frac{1}{2}}, y, z, t), \mathbf{q}_h^+(x_{i+\frac{1}{2}}, y, z, t) \right) dz dy dt, . \end{aligned}$$

- **Reconstruction:** introduce a **nodal basis** of polynomials $\psi_l(\xi)$ of order M defined with respect to a set of Gauss-Legendre nodal points λ_k , such that $\psi_l(\lambda_k) = \delta_{lk}$



- Build **one-dimensional reconstruction stencils** along each direction

$$\mathcal{S}_{ijk}^{s,x} = \bigcup_{e=i-L}^{i+R} l_{ejk}, \quad \mathcal{S}_{ijk}^{s,y} = \bigcup_{e=j-L}^{j+R} l_{iek}, \quad \mathcal{S}_{ijk}^{s,z} = \bigcup_{e=k-L}^{k+R} l_{ije},$$



- three stencils for even M
- four stencils for odd M
- $M + 1$ cells in each stencil

- ① Perform an entire polynomial **WENO reconstruction** along x - **direction**:

$$\mathbf{w}_h^{s,x}(x, t^n) = \sum_{p=0}^M \psi_p(\xi) \hat{\mathbf{w}}_{ijk,p}^{n,s} := \psi_p(\xi) \hat{\mathbf{w}}_{ijk,p}^{n,s} \quad \forall \quad \mathcal{S}_{ijk}^{s,x}$$

Impose integral conservation on all elements of the stencil:

$$\frac{1}{\Delta x_e} \int_{x_{e-\frac{1}{2}}}^{x_{e+\frac{1}{2}}} \psi_p(\xi(x)) \hat{\mathbf{w}}_{ijk,p}^{n,s} dx = \mathbf{u}_{ejk}^n, \quad \forall l_{ejk} \in \mathcal{S}_{ijk}^{s,x}$$

Perform a data-dependent nonlinear combination:

$$\mathbf{w}_h^x(x, t^n) = \psi_p(\xi) \hat{\mathbf{w}}_{ijk,p}^n, \quad \text{with} \quad \hat{\mathbf{w}}_{ijk,p}^n = \sum_{s=1}^{N_s} \omega_s \hat{\mathbf{w}}_{ijk,p}^{n,s}$$

$$\omega_s = \frac{\tilde{\omega}_s}{\sum_k \tilde{\omega}_k}, \quad \tilde{\omega}_s = \frac{\lambda_s}{(\sigma_s + \epsilon)^r}$$

$$\sigma_s = \sum_{pm} \hat{\mathbf{w}}_{ijk,p}^{n,s} \hat{\mathbf{w}}_{ijk,m}^{n,s} \quad \Sigma_{pm} = \sum_{\alpha=1}^M \int_0^1 \frac{\partial^\alpha \psi_p(\xi)}{\partial \xi^\alpha} \cdot \frac{\partial^\alpha \psi_m(\xi)}{\partial \xi^\alpha} d\xi.$$

[see Dumbser & Käser, (2007), JCP, 221, 693]

- 2 Perform a second polynomial **WENO reconstruction** along y - **direction** using as input the $M + 1$ degrees of freedom $\hat{\mathbf{w}}_{ijk,p}^n$:

$$\mathbf{w}_h^{s,y}(x, y, t^n) = \psi_p(\xi)\psi_q(\eta)\hat{\mathbf{w}}_{ijk,pq}^{n,s}.$$

Apply integral conservation is now in the y direction:

$$\frac{1}{\Delta y_e} \int_{y_e - \frac{1}{2}}^{y_e + \frac{1}{2}} \psi_q(\eta(y)) \hat{\mathbf{w}}_{ijk,pq}^{n,s} dy = \hat{\mathbf{w}}_{iek,p}^n, \quad \forall iek \in \mathcal{S}_{ijk}^{s,y}.$$

\Rightarrow

$$\mathbf{w}_h^y(x, y, t^n) = \psi_p(\xi)\psi_q(\eta)\hat{\mathbf{w}}_{ijk,pq}^n, \quad \text{with} \quad \hat{\mathbf{w}}_{ijk,pq}^n = \sum_{s=1}^{N_s} \omega_s \hat{\mathbf{w}}_{ijk,pq}^{n,s},$$

- 3 Perform the last polynomial **WENO reconstruction** along z - **direction** using as input the $(M + 1)^2$ degrees of freedom $\hat{\mathbf{w}}_{ijk,pq}^n$: We therefore have

$$\mathbf{w}_h^{s,z}(x, y, z, t^n) = \psi_p(\xi)\psi_q(\eta)\psi_r(\zeta)\hat{\mathbf{w}}_{ijk,pqr}^{n,s}.$$

with the integral conservation written as above,

$$\frac{1}{\Delta z_e} \int_{z_e - \frac{1}{2}}^{z_e + \frac{1}{2}} \psi_r(\zeta(z)) \hat{\mathbf{w}}_{ijk,pqr}^{n,s} dz = \hat{\mathbf{w}}_{iek,pq}^n, \quad \forall ije \in \mathcal{S}_{ijk}^{s,z}.$$

\implies The final three-dimensional WENO polynomial

$$\mathbf{w}_h(\mathbf{x}, t^n) = \psi_p(\xi)\psi_q(\eta)\psi_r(\zeta)\hat{\mathbf{w}}_{ijk,pqr}^n, \quad \hat{\mathbf{w}}_{ijk,pqr}^n = \sum_{s=1}^{N_s} \omega_s \hat{\mathbf{w}}_{ijk,pqr}^{n,s}.$$

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Local space-time DG predictor

[see Dumbser et al. (2008), JCP, 227, 3971]

An alternative to the Cauchy-Kovalewski procedure to obtain the time evolution of the reconstructed polynomials and build one-step time-update numerical schemes.

In practice we need

$$\mathbf{q}_h \implies \mathbf{f}_{i+\frac{1}{2},jk} = \frac{1}{\Delta t} \frac{1}{\Delta y_j} \frac{1}{\Delta z_k} \int_{t^n}^{t^{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \tilde{\mathbf{f}}(\mathbf{q}_h^-(x_{i+\frac{1}{2}}, y, z, t), \mathbf{q}_h^+(x_{i+\frac{1}{2}}, y, z, t)) dz dy, dt,$$

$$\frac{\partial \mathbf{u}}{\partial \tau} + \frac{\partial \mathbf{f}^*}{\partial \xi} + \frac{\partial \mathbf{g}^*}{\partial \eta} + \frac{\partial \mathbf{h}^*}{\partial \zeta} = \mathbf{S}^*$$

with

$$\mathbf{f}^* = \frac{\Delta t}{\Delta x_i} \mathbf{f}, \quad \mathbf{g}^* = \frac{\Delta t}{\Delta y_j} \mathbf{g}, \quad \mathbf{h}^* = \frac{\Delta t}{\Delta z_k} \mathbf{h}, \quad \mathbf{S}^* = \Delta t \mathbf{S}.$$

We then multiply by the test function $\theta_p(\boldsymbol{\xi}, \tau)$ and integrate in space-time

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta_q \left(\frac{\partial \mathbf{u}}{\partial \tau} + \frac{\partial \mathbf{f}^*}{\partial \xi} + \frac{\partial \mathbf{g}^*}{\partial \eta} + \frac{\partial \mathbf{h}^*}{\partial \zeta} - \mathbf{S}^* \right) d\xi d\eta d\zeta d\tau = 0.$$

where we choose a tensor product of the basis functions

$$\theta_p(\boldsymbol{\xi}, \tau) = \psi_p(\xi) \psi_q(\eta) \psi_r(\zeta) \psi_s(\tau).$$

Integration by parts in time yields...

$$\begin{aligned}
 & \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 1) \mathbf{u}(\boldsymbol{\xi}, 1) d\xi d\eta d\zeta - \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\frac{\partial}{\partial \tau} \theta_q \right) \mathbf{u} d\xi d\eta d\zeta d\tau \\
 & + \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left[\theta_q \left(\frac{\partial \mathbf{f}^*}{\partial \xi} + \frac{\partial \mathbf{g}^*}{\partial \eta} + \frac{\partial \mathbf{h}^*}{\partial \zeta} - \mathbf{S}^* \right) \right] d\xi d\eta d\zeta d\tau \\
 & = \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 0) \mathbf{w}_h(\boldsymbol{\xi}, t^n) d\xi d\eta d\zeta.
 \end{aligned}$$

We introduce the discrete space-time solution \mathbf{q}_h

$$\mathbf{q}_h = \mathbf{q}_h(\boldsymbol{\xi}, \tau) = \theta_p(\boldsymbol{\xi}, \tau) \hat{\mathbf{q}}_p,$$

and similarly for the fluxes and sources

$$\mathbf{f}_h^* = \theta_p \hat{\mathbf{f}}_p^* = \theta_p \mathbf{f}^*(\hat{\mathbf{q}}_p), \quad \dots \quad \mathbf{S}_h^* = \theta_p \hat{\mathbf{S}}_p^* = \theta_p \mathbf{S}^*(\hat{\mathbf{q}}_p).$$

Insert everything in Eq. above and obtain....

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 1) \theta_p(\boldsymbol{\xi}, 1) \hat{\mathbf{q}}_p \, d\xi d\eta d\zeta - \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\frac{\partial}{\partial \tau} \theta_q \right) \theta_p \hat{\mathbf{q}}_p \, d\xi d\eta d\zeta d\tau \\
& + \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left[\theta_q \left(\frac{\partial}{\partial \xi} \theta_p \hat{\mathbf{f}}_p^* + \frac{\partial}{\partial \eta} \theta_p \hat{\mathbf{g}}_p^* + \frac{\partial}{\partial \zeta} \theta_p \hat{\mathbf{h}}_p^* - \theta_p \hat{\mathbf{S}}_p^* \right) \right] d\xi d\eta d\zeta d\tau \\
& = \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 0) \mathbf{w}_h(\boldsymbol{\xi}, t^n) \, d\xi d\eta d\zeta,
\end{aligned}$$

which is a nonlinear algebraic equation system for the unknown coefficients $\hat{\mathbf{q}}_p$.

In a more compact form:

$$\mathbf{K}_{qp}^1 \hat{\mathbf{q}}_p + \mathbf{K}_{qp}^\xi \cdot \hat{\mathbf{f}}_p^* + \mathbf{K}_{qp}^\eta \hat{\mathbf{g}}_p^* + \mathbf{K}_{qp}^\zeta \hat{\mathbf{h}}_p^* = \mathbf{M}_{qp} \hat{\mathbf{S}}_p^* + \mathbf{F}_{qm}^0 \hat{\mathbf{w}}_m^n,$$

with the various matrices defined as

$$\mathbf{K}_{qp}^1 = \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 1) \theta_p(\boldsymbol{\xi}, 1) d\boldsymbol{\xi} - \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\frac{\partial}{\partial \tau} \theta_q \right) \theta_p d\boldsymbol{\xi} d\tau,$$

$$\mathbf{K}_{qp}^\xi = \left(\mathbf{K}_{qp}^\xi, \mathbf{K}_{qp}^\eta, \mathbf{K}_{qp}^\zeta \right) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta_q \frac{\partial}{\partial \xi} \theta_p d\boldsymbol{\xi} d\tau,$$

$$\mathbf{M}_{qp} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \theta_q \theta_p d\boldsymbol{\xi} d\tau, \quad \mathbf{F}_{qm}^0 = \int_0^1 \int_0^1 \int_0^1 \theta_q(\boldsymbol{\xi}, 0) \psi_m(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

The product of the matrices with the vectors of degrees of freedom can be efficiently implemented in a dimension-by-dimension manner. An iterative scheme can be adopted [see Dumbser & Zanotti (2009), JCP, 228, 6991]

$$\mathbf{K}_{qp}^1 \hat{\mathbf{q}}_p^{k+1} - \mathbf{M}_{qp} \hat{\mathbf{S}}_p^{*,k+1} = \mathbf{F}_{qm}^0 \hat{\mathbf{w}}_m^n - \mathbf{K}_{qp}^\xi \cdot \hat{\mathbf{f}}_p^{*,k} - \mathbf{K}_{qp}^\eta \hat{\mathbf{g}}_p^{*,k} - \mathbf{K}_{qp}^\zeta \hat{\mathbf{h}}_p^{*,k}$$

Once this is done, we have everything to write the scheme as

$$\begin{aligned} \mathbf{u}_{ijk}^{n+1} &= \mathbf{u}_{ijk}^n - \frac{\Delta t}{\Delta x_i} \left(\mathbf{f}_{i+\frac{1}{2},jk} - \mathbf{f}_{i-\frac{1}{2},jk} \right) - \frac{\Delta t}{\Delta y_j} \left(\mathbf{g}_{i,j+\frac{1}{2},k} - \mathbf{g}_{i,j-\frac{1}{2},k} \right) \\ &\quad - \frac{\Delta t}{\Delta z_k} \left(\mathbf{h}_{ij,k+\frac{1}{2}} - \mathbf{h}_{ij,k-\frac{1}{2}} \right) + \Delta t \mathbf{S}_{ijk}, \end{aligned}$$

- Local Lax-Friedrichs flux

$$\tilde{\mathbf{f}}(\mathbf{q}_h^-, \mathbf{q}_h^+) = \frac{1}{2} (\mathbf{f}(\mathbf{q}_h^-) + \mathbf{f}(\mathbf{q}_h^+)) - \frac{1}{2} |\mathbf{s}_{\max}| (\mathbf{q}_h^+ - \mathbf{q}_h^-), \quad (1)$$

where $|\mathbf{s}_{\max}|$ denotes the maximum absolute value of the eigenvalues of the Jacobian matrix $\mathbf{A} = \partial \mathbf{f} / \partial \mathbf{u}$.

- Osher-type flux

$$\tilde{\mathbf{f}}(\mathbf{q}_h^-, \mathbf{q}_h^+) = \frac{1}{2} (\mathbf{f}(\mathbf{q}_h^-) + \mathbf{f}(\mathbf{q}_h^+)) - \frac{1}{2} \left(\int_0^1 |\mathbf{A}(\psi(s))| ds \right) (\mathbf{q}_h^+ - \mathbf{q}_h^-), \quad (2)$$

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AMR implementation

We have developed a **cell-by-cell** AMR technique in which the computational domain is discretized with a uniform Cartesian grid at the coarsest level.

- Adopt a refinement criterion, marking a cell for refinement if $\chi_m > \chi_{\text{ref}}$, where

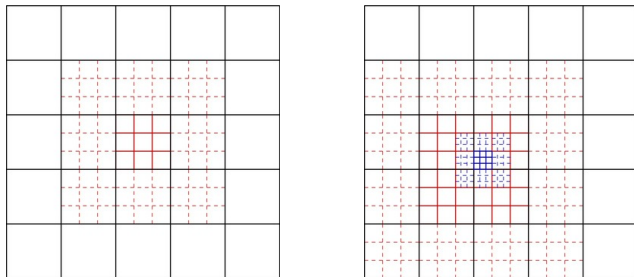
$$\chi_m = \sqrt{\frac{\sum_{k,l} (\partial^2 \Phi / \partial x_k \partial x_l)^2}{\sum_{k,l} [(|\partial \Phi / \partial x_k|_{i+1} + |\partial \Phi / \partial x_k|_i) / \Delta x_l + \varepsilon |(\partial^2 / \partial x_k \partial x_l)| |\Phi|]^2}}.$$

- When a cell of the level ℓ is refined, it is subdivided as

$$\Delta x_\ell = \tau \Delta x_{\ell+1} \quad \Delta y_\ell = \tau \Delta y_{\ell+1} \quad \Delta z_\ell = \tau \Delta z_{\ell+1} \quad \Delta t_\ell = \tau \Delta t_{\ell+1}$$

- Each cell \mathcal{C}_m , at any level of refinement, has one among three possible *status* flags.
 - *active cell*
 - *virtual child cell*
 - *virtual mother cell*

...AMR implementation



- If the Voronoi neighbors of an active refined cell C_m are not themselves at the same level of refinement of C_m , they have virtual children at the same level of refinement of C_m .
- In order to keep the reconstruction local on the coarser grid level, we have $\tau \geq M$.
- The levels of refinement of two cells that are Voronoi neighbors of each other can only differ by at most unity.

- **The beauty of the local-spacetime DG predictor:** it does not need any exchange of information with neighbor elements, even if two adjacent cells are on different levels of refinement.

- **Projection**

Projection is the typical AMR operation, by which an active mother assigns values to the virtual children ($\sigma = 1$) at intermediate times

$$\bar{\mathbf{u}}_m(t_\ell^n) = \frac{1}{\Delta x_\ell} \frac{1}{\Delta y_\ell} \frac{1}{\Delta z_\ell} \int_{C_m} \mathbf{q}_h(\mathbf{x}, t_\ell^n) d\mathbf{x}. \quad (3)$$

Needed for performing the reconstruction on the finer grid level at intermediate times.

- **Averaging**

Averaging is another typical AMR operation by which a virtual mother cell ($\sigma = -1$) obtains its cell average by averaging recursively over the cell averages of all its children at higher refinement levels.

$$\bar{\mathbf{u}}_m = \frac{1}{\mathfrak{r}^d} \sum_{C_k \in \mathcal{B}_m} \bar{\mathbf{u}}_k. \quad (4)$$

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Local Timestepping

Every refinement level is advanced in time with its local timestep $\Delta t_\ell = \tau \Delta t_{\ell+1}$. **Update criterion:**

$$t_\ell^{n+1} \leq t_{\ell-1}^{n+1}, \quad 0 \leq \ell \leq \ell_{\max}, \quad (5)$$

Starting from the common initial time $t = 0$, the finest level of refinement ℓ_{\max} is evolved first and performs a number of τ sub-timesteps before the next coarser level $\ell_{\max} - 1$ performs its first time update.

\Rightarrow A total amount of τ^ℓ sub-timesteps on each level are performed in order to reach the time t_0^{n+1} of the coarsest level.

RHD equations

$$\partial_t \mathbf{u} + \partial_i \mathbf{f}^i = 0,$$

The conservative variables and the corresponding fluxes in the i direction are

$$\mathbf{u} = \begin{bmatrix} D \\ S_j \\ E \end{bmatrix}, \quad \mathbf{f}^i = \begin{bmatrix} v^i D \\ W_j^i \\ S^i \end{bmatrix}. \quad (6)$$

$$D = \rho\gamma,$$

$$S_i = \rho h \gamma^2 v_i,$$

$$E = \rho h \gamma^2 - p,$$

where $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor of the fluid with respect to the laboratory observer and

$$W_{ij} \equiv \rho h \gamma^2 v_i v_j + p \delta_{ij}$$

Just one test...

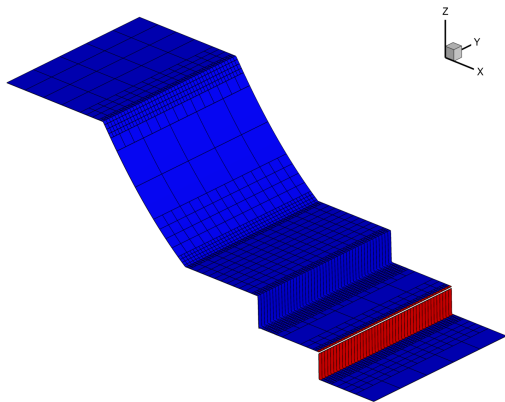


Figura : Standard Riemann problem: the shock wave is resolved within one single cell!!

The Richtmyer–Meshkov instability

The Richtmyer–Meshkov (RM) instability is a typical fluid instability which develops when a shock wave crosses a contact discontinuity within a fluid, or between two different fluids.

Relevant in

- Inertial Confinement Fusion
- Supernovae remnant formation
- Relativistic jets

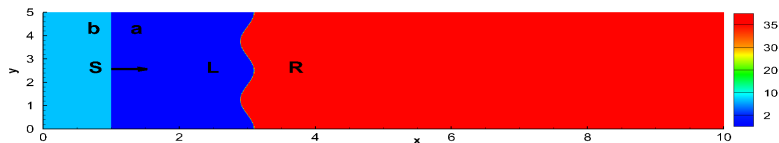


Figura : Schematic representation of the initial conditions in a representative model with Atwood number $A > 0$.

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- High Order in combination with AMR is extremely beneficial, in particular when small scale structures need to be solved

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- We have presented the first ADER-WENO finite volume scheme for relativistic hydrodynamics on AMR grids
- Compared to Runge–Kutta time stepping, the use of a high order one–step scheme in time reduces the number of nonlinear WENO reconstructions and the number of necessary MPI communications.
- High Order in combination with AMR is extremely beneficial, in particular when small scale structures need to be solved
- When applied to the study of the Richtmyer–Meshkov instability, new interesting results are obtained