Hydrodynamical Escape Approximation for Pluto

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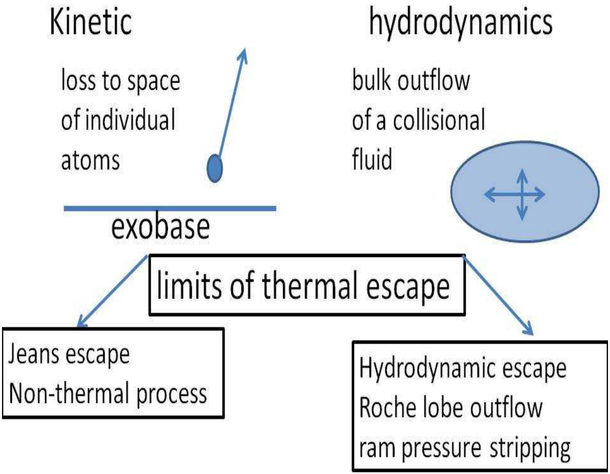
Outline

- Hydrodynamic Escape and the Corresponding Equations
- Boundary Conditions
- Relaxation Methods
- Previous Results
- Our Numerical Results
- A Comparison Study
- An Application on Isotope
- Conclusion

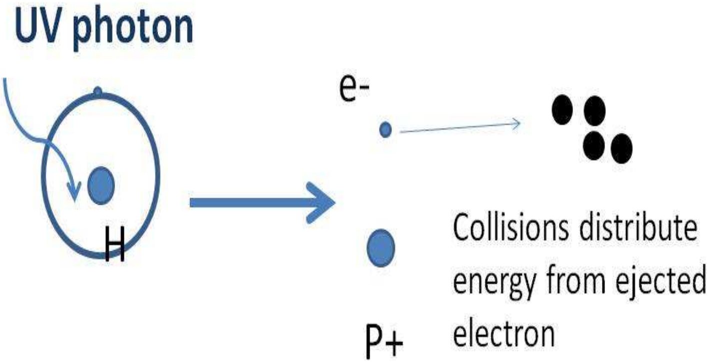
HD 209458

Artist's illustration of the hot Jupiter-like planet orbiting star HD 209458. Astronomers have now discovered oxygen and carbon in in a violent atmospheric 'blow-off.' (Image credit: Alfred Vidal-Madjar)

Two Classes of Escape Mechanism
Thermal or Non-Thermal



UV photon heat the upper atmosphere



Absorption of Solar EUV/UV

Solar UV absorption drives atmospheric:

- constituent densities,
- thermal structure, and
- dynamics

Solar UV is absorbed by

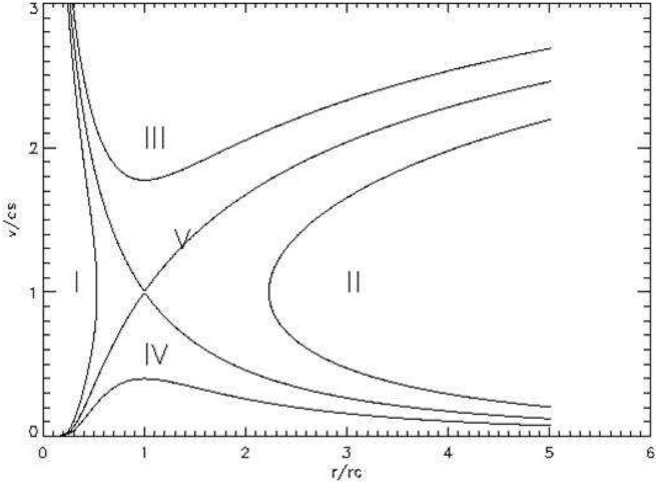
- ozone (200-320 nm)
- molecular oxygen (140-242 nm)

Isothermal Atmosphere (Hydrostatic)

$$\begin{aligned}
\frac{dp}{dz} &= -\rho g \\
p = nkT &= \rho \frac{k}{m} T, \quad \rho = mp/kT \\
\frac{dp}{p} = -\frac{mg}{kT} dz &\equiv -\frac{dz}{H_a} \\
\ln p &= -\frac{z}{H_a} + \text{const.} \\
p &= p_0 \exp\left(-\frac{z - z_0}{H_a}\right)
\end{aligned}$$

As $z \to \infty$, $p \to 0$ as expected.

Parker Isothermal Solution



$$u = c_s, \quad r = r_s = \frac{GM_s}{2c_s^2}$$

Gravity Varies with Height

$$
\begin{aligned}
g &= \frac{GM}{r^2} \\
\frac{dp}{p} &= -\frac{GMm}{r^2kT}dr \\
\ln p &= \frac{GMm}{rkT} + \text{const.} \\
p &= p_0 \exp(\frac{GMm}{kT}(\frac{1}{r} - \frac{1}{r_0})).
\end{aligned}
$$

As $r \to \infty$, $p \to p_0 > 0 \Rightarrow$ infinite mass.

Modifications

- The atmosphere becomes collisionless at some height, so that pressure is not defined in the normal manner.
- The atmosphere is not hydrostatic, i.e. it must expand into space.

Hydrodynamic Escape Equations

$$\frac{\partial(\rho r^2)}{\partial t} + \frac{\partial(\rho u r^2)}{\partial r} = 0, \qquad (1)$$

$$\frac{\partial(\rho u r^2)}{\partial t} + \frac{\partial(\rho u^2 r^2 + p r^2)}{\partial r} = -\rho G M + 2pr, \qquad (2)$$

$$\frac{\partial(E r^2)}{\partial t} + \frac{\partial((E+p) u r^2)}{\partial r} = -\rho u G M + q r^2 + \frac{\partial}{\partial r}(\kappa r^2 \frac{\partial T}{\partial r}), \qquad (3)$$

where the total energy E, internal energy e, and the pressure p are given by $E = \rho(\frac{u^2}{2} + e)$, $\quad e = \frac{p}{\rho(\gamma-1)}$, $\quad p = \rho R T$. Here, heating (q) from the central star and thermal conduction (coefficient $\kappa = \nu T$ are also imposed, where ν is a constant) in the atmosphere are also considered.

A Mathematical Problem

$$\frac{du}{dr} = \frac{u}{u^2 - \gamma k_B T/\mu m_e} \times [2\gamma k_B T/(\mu m_e r) - (\gamma - 1)q/(\rho u) - \frac{GM_p}{r^2} - \frac{\gamma - 1}{\rho u r^2}\frac{d}{dr}(\kappa r^2 \frac{dT}{dr})\Big] \quad (4)$$

$$\frac{d\rho}{dr} = -\frac{\rho}{u}\frac{du}{dr} - \frac{2\rho}{r} \quad (5)$$

$$\frac{dT}{dr} = (\gamma - 1)\left(\frac{q}{\rho u}\frac{\mu m_e}{k_B} + \frac{T}{\rho}\frac{d\rho}{dr} + \frac{1}{r^2 \rho u}\frac{\mu m_e}{k_B}\frac{d}{dr}(\kappa r^2 \frac{dT}{dr})\right) \quad (6)$$

Boundary Conditions

We impose the density ρ_b and the temperature T_b at the bottom boundary r_b,

$$\rho_b = \rho(r_b, t), \quad T_b = T(r_b, t). \tag{7}$$

There two conditions are required. The transonic solution is desired, and it follows that

$$u^2 = c^2 = \gamma RT, \tag{8}$$

and

$$2\gamma RT/r - (\gamma - 1)q/(\rho u) - \frac{GM_p}{r^2} - \frac{\gamma - 1}{\rho u r^2}\frac{d}{dr}(\kappa r^2 \frac{dT}{dr}) = 0 \tag{9}$$

at the sonic position r_s.

More Boundary Conditions

We introduce three extra dependent variables

$$L = -\kappa r^2 \frac{dT}{dr}, \quad W = \frac{dL}{dr}, \text{ and } Z = r_s - r_b.$$

$$L = -\kappa r^2(\gamma - 1)\left[T(-\frac{1}{u}\frac{du}{dr}\Big|_{r_s} - \frac{2}{r}) - \frac{W\mu m_e}{\rho u r^2 k_B}\right], \tag{10}$$

- The mathematical problem is well-posed.
- The existence and uniqueness of the solution of the HEP is un-resolved.

Relaxation Method

The quantities (T, u, ρ, Z, L, W) are given at discrete points:

$$\begin{aligned}
\frac{du}{dr} &= \frac{u}{u^2 - \gamma k_B T/\mu m_e} \times \\
&\quad \left[2\gamma k_B T/(\mu m_e r) - (\gamma - 1)q/(\rho u) - \frac{GM_p}{r^2} + \frac{\gamma - 1}{\rho u r^2} W\right] \\
\frac{d\rho}{dr} &= -\frac{\rho}{u}\frac{du}{dr} - \frac{2\rho}{r} \\
\frac{dT}{dr} &= -\frac{L}{\kappa r^2} \\
\frac{dL}{dr} &= \frac{\rho u r^2 k_B}{\mu m_e}\left[-\frac{1}{\gamma - 1}\frac{dT}{dr} + \frac{T}{\rho}\frac{d\rho}{dr} + \frac{\mu m_e}{\rho u k_B} q\right] \\
\frac{dL}{dr} &= W, \quad \frac{dZ}{dr} = 0
\end{aligned}$$

The Derivative of Velocity at the Sonic Position

$$\frac{du}{dr}|_{r_s} = \frac{\gamma - 1}{\gamma + 1} \left\{ -\frac{2u}{r} - \frac{\gamma W}{2\rho u^2 r^2} + \frac{1}{2u} \left[(\frac{\gamma W}{\rho u r^2})^2 - 8\frac{(\gamma + 1)}{(\gamma - 1)} \frac{uW}{\rho r^3} + \frac{16}{(\gamma - 1)} \frac{uW}{\rho r^3} + \frac{8(5 - 3\gamma)}{(\gamma - 1)^2} \frac{u^4}{r^2} \right]^{1/2} \right\},$$

Darrell F. Strobel, 2008, Icarus

614 D.F. Strobel / Icarus 193 (2008) 612–619

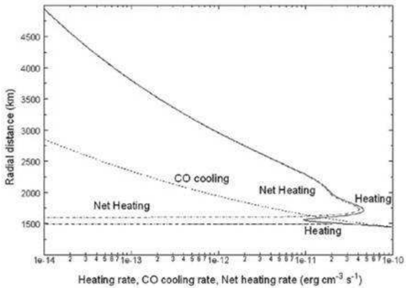
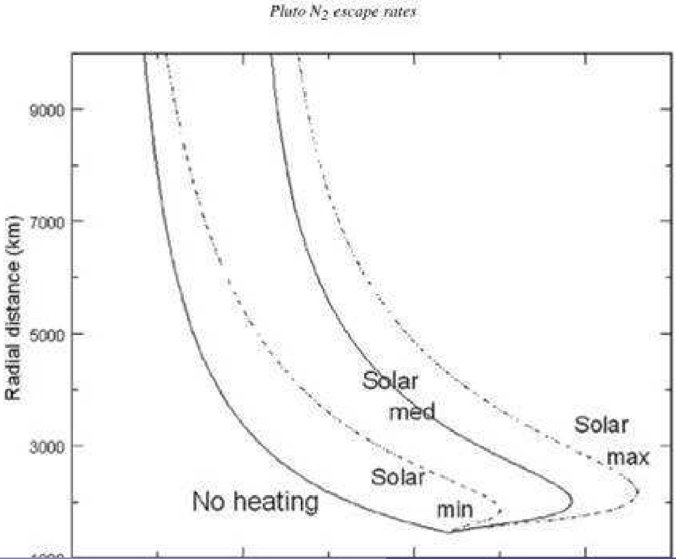


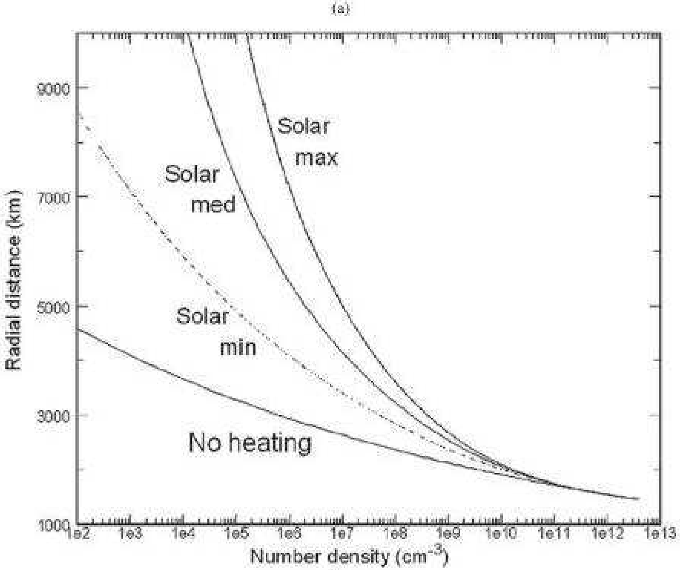
Fig. 1. The combined N_2 and CH_4 heating rate, the CO cooling rate, and the net heating rate for solar medium conditions at 30 AU in units of erg cm^{-3} s^{-1}.

- The number density 4E+12 ($1/cm^3$) and the temperature 88.2 K at the lower boundary.
- Conduction $\kappa(T) = 8.37T$ and $M_s = 1.305E + 25$ g.

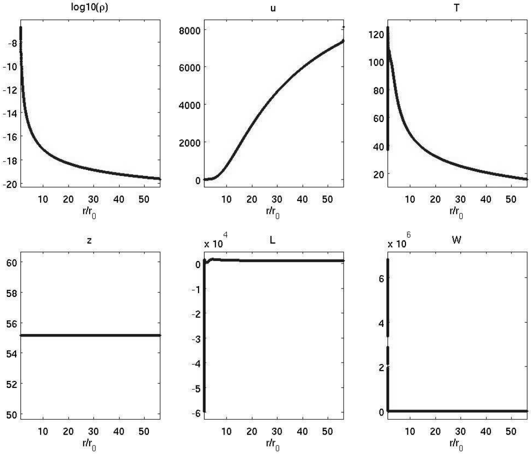
Results of Strobel



Results of Strobel



Results of the Relaxation Method



Hybrid Fluid/Kinetic Models

The hybrid fluid/kinetic model of Erwin, Tucker, and Johnson (2012) attempts to describe the atmospheric escape more realistically. Such an approach consists of the hydrodynamic escape result of Strobel (2008) and a modified Maxwell-Boltzmann to include the bulk velocity u in the velocity distribution (Yell,2004; Tian et al., 2009; Volkov et al., 2011). However, the treatment of the transition from the collisional regime to the non collisional regime is approximate.

Jeans escape

Jeans escape corresponds to the literal evaporation atom by atom, molecule by molecule, off the top of the atmosphere. At lower altitudes, collisions confine particles, but above a certain altitude, known as the exobase, air is so tenuous that gas particles hardly ever collide.

The Jean's escape framework in terms of the Jeans parameter, $\lambda(r) = GM\mu m_p / rk_B T$. The Jean's escape rate is given as

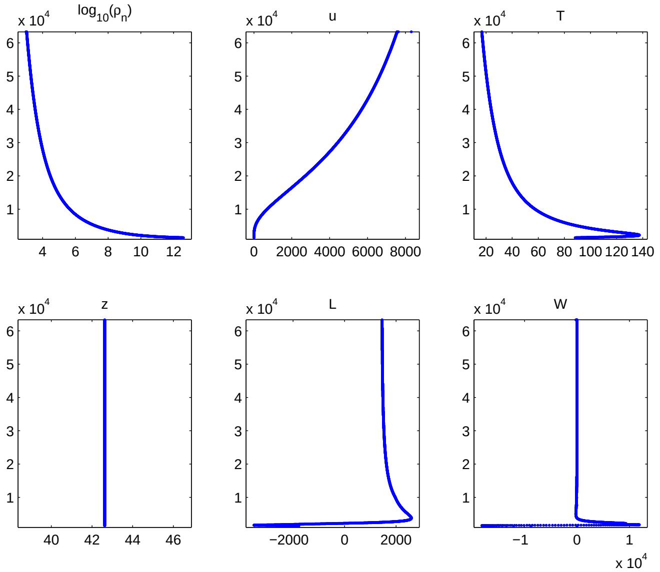
$$\Phi_J = 4\pi r_x^2 n_x \sqrt{\frac{kT_x}{2\pi\mu m_p}} (1 + \lambda_x) \exp(-\lambda_x).$$

Settings for Pluto Simulations

- The lower boundary $r_{\text{min}} = 1.45E + 8\ cm$
- The temperature $T = 88.2$ at the lower boundary
- The number density $n = 4E + 12$ at the lower boundary
- The mass of the Pluto $M_p = 1.31E + 25\ g$
- For nitrogen simulation, $\mu = 28$.

We request the heat profile q of solar mean from Strobel for two numerical simulations.

Results for Pluto Simulations



Models

- Model A is a consistent model.
- Model B is a fixed heat profile in Strobel (2009) to compare with Strobel's result.
- Model C is a fixed heat profile to compare with Johnson's simulations.

Comparision

Model unit	T_{pk} K	$r(T_{pk})$ 10^8 cm	$\dot{M}$ 10^{27}cm^{-3}	Q erg$/s$
A	137	2.20	3.98	16.06
B	132	2.08	1.73	11.35
C	123	2.07	0.77	7.8
Strobel	117	2.02	3.26	11.35
Johnson	120	2.13	2.58	7.8

This table shows us the peak temperature T_{pk}, its location $r(T_{pk})$, the mass flux $\dot{M}$, and the total heat Q for the model A, B, C, Strobel and Johnson's simulations.

Comparison

Model unit	$r(T_{exo})$ 10^8 cm	ρ 10^8cm^{-3}	T_{exo} K	$\lambda(r(T_{exo}))$
A	7.93	0.014	67	5.6
B	7.20	0.015	68	6.2
C	6.20	0.014	71	6.7
Strobel	4.83	0.027	66	9.3
Johnson	7.40	0.002	80	3.8

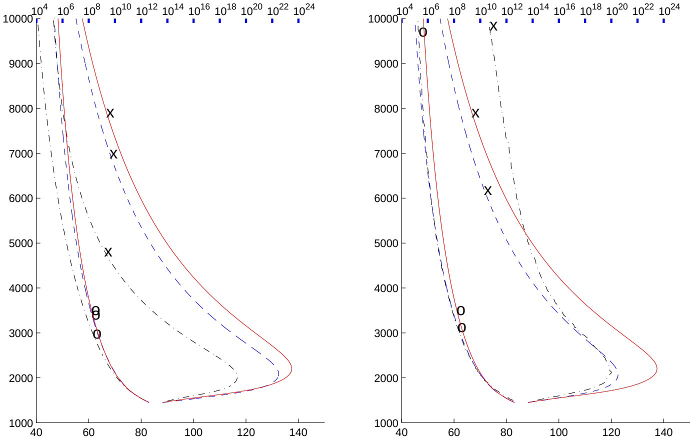
This table shows us the exobases for temperature and density and their corresponding temperature, density and the Jeans parameter for Model A, B, C, Strobel and Johnson.

Comparison

Model unit	$r(\rho_{exo})$	ρ_{exo} 10^8cm^{-3}	T K	$\lambda(r(\rho_{exo}))$
A	3.5	1.55	112	7.5
B	3.4	1.62	106	8.2
C	3.2	1.86	105	9.0
Strobel	3.0	1.97	95	10.4
Johnson	9.7	3.11	74	4.1

This table shows us the exobases for temperature and density and their corresponding temperature, density and the Jeans parameter for Model A, B, C, Strobel and Johnson.

Comparison of results for Pluto's atmosphere



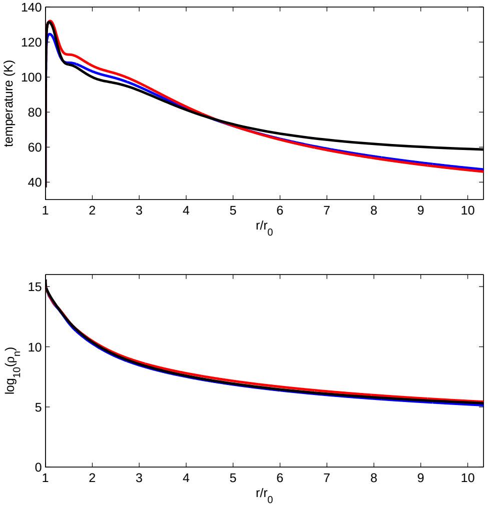
The superpose profiles of density and temperature for the models A, B, Strobel, and A, C, Johnson labeled by red, blue and black line are shown in left and right, respectively. The marker "x" and "o" indicate the exobase locations.

New Settings for Pluto Simulations

- The lower boundary $r_{\text{min}} = 1.18E + 8$ *cm*
- The temperature $T = 37$ at the lower boundary
- The number density $T = 4E + 15$ at the lower boundary
- The mass of the Pluto $M_p = 1.31E + 25$ *g*
- For nitrogen simulation, $\mu = 28$.

We request the heat profile q of solar mean from Strobel for two numerical simulations.

Comparison of results for Pluto's atmosphere



Notations for isotope simulations

Let us denote the density, velocity, and temperature for the major species as ρ, u, and T, respectively; the corresponding quantities for the minor constituent are labeled with the subscript "i." The temperature T, the velocity u, and the density ρ are fixed for the simulation of the minor species.

Equations for isotope simulations

Following (Zahnle and Kasting 1986), they are described as

$$\begin{aligned} \frac{d}{dr}(\rho_i u_i r^2) &= 0, & (11) \\ \frac{d}{dr}(\rho_i u_i^2 r^2 + p_i r^2) &= -GM_p\rho_i + 2p_i r \\ &+ \rho_2 \nu_i (u - u_i) r^2 \Phi_i & (12) \end{aligned}$$

where $\Phi_i = 1$, $\nu_i = \sigma nc = \sigma n \sqrt{\gamma k_B T / \mu m_p}$ and $p_i = \rho_i k_B T / \mu_i m_p$.

Approximation of isotope solutions

Under the assumption of the temperature of the major and minor species is the same, $T = T_i$, the mass and momentum equations for minor species are only needed to be solved. Equations (11) and (12) can be rewritten as

$$\frac{du_i}{dr} = \frac{u_i}{u_i^2 - k_B T/\mu_i m_p} \times \left[-\frac{k_B}{\mu_i m_p}\frac{dT}{dr} + \frac{2k_B T}{\mu_i m_p r} - \frac{GM_p}{r^2} + \nu_i (u - u_i)\Phi_i \right] \quad (13)$$

$$\frac{d\rho_i}{dr} = -\frac{\rho_i}{u_i}\frac{du_i}{dr} - \frac{2\rho_i}{r} \quad (14)$$

- The singularity occurs at the thermal critical point.

Thermal critical

Since the solution is required to be smooth and (13), the velocity u_i at the singular position r_c should satisfy

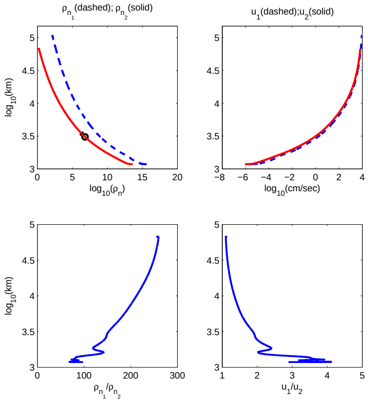
$$u_i^2 = \frac{k_B T}{\mu_i m_p} \tag{15}$$

and

$$-\frac{k_B}{\mu_i m_p}\frac{dT}{dr} + \frac{2k_B T}{\mu_i m_p r_c} - \frac{GM_p}{r^2} + \nu_i(u - u_i)\Phi_i = 0. \tag{16}$$

In the other words, the numerator and denomerator in (13) are vanished. This singular position r_c is identified as a thermal critical and also regarded as the top boundary.

Results for isotope simulation



$$\dot{M} = 1.4702 \times 10^{26}\text{cm}^{-3} \text{ and } \dot{M}_i = 5.7139 \times 10^{23}\text{cm}^{-3}$$

Conclusion

- The mathematical study of the hydrodynamic escape problem is in its infancy, since even the existence and uniqueness of solutions are still open. On the other hand, through our numerical simulations, the hydrodynamic solution can approximate the situation of the rare gas for Pluto.
- Under the framework of hydrodynamic approximation, it was generally accepted that the process produced rather small isotopic fractionation. Here, we show that the escape highly fractionates the isotopic composition of nitrogen.

Thanks for your attention

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