Modifications for improved accuracy on angular momentum in black hole evolution

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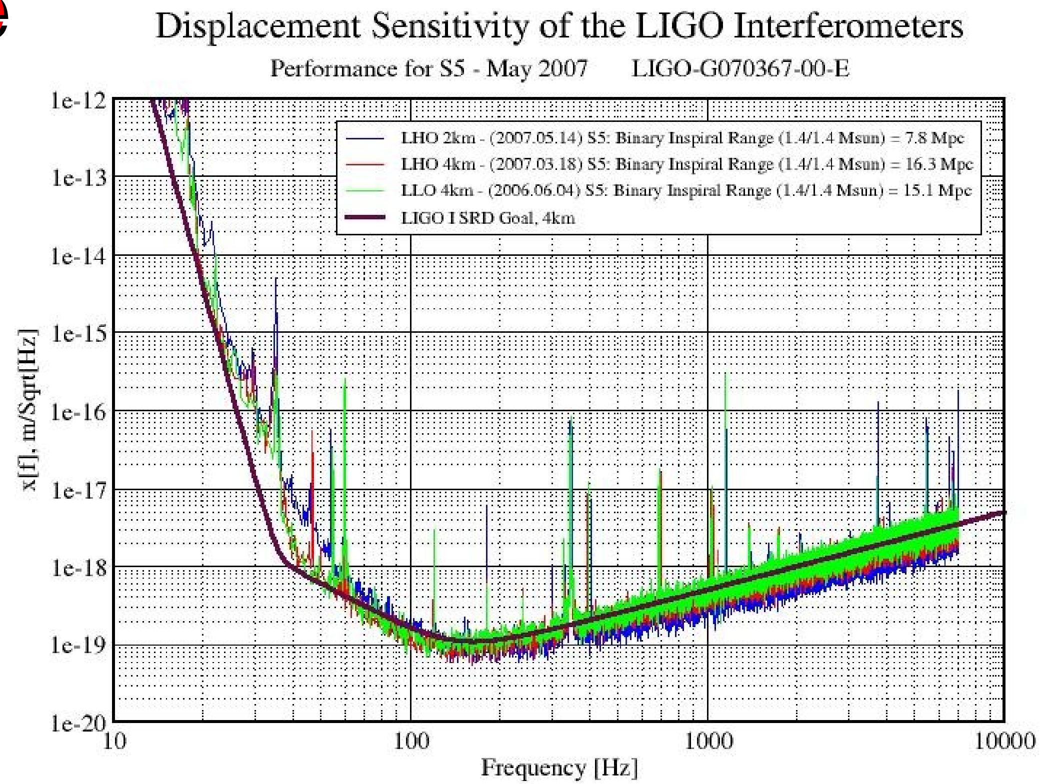
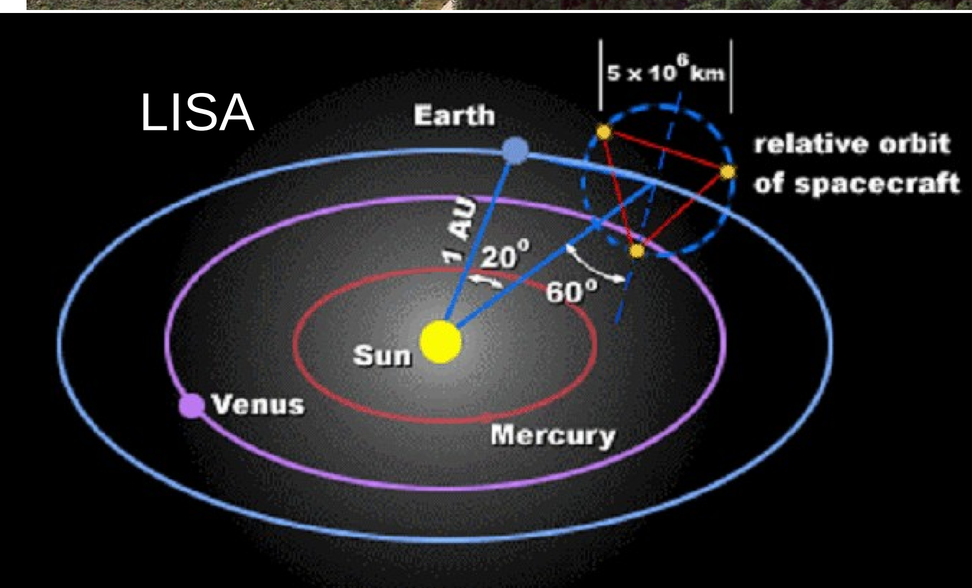
Gravitational Wave Detection

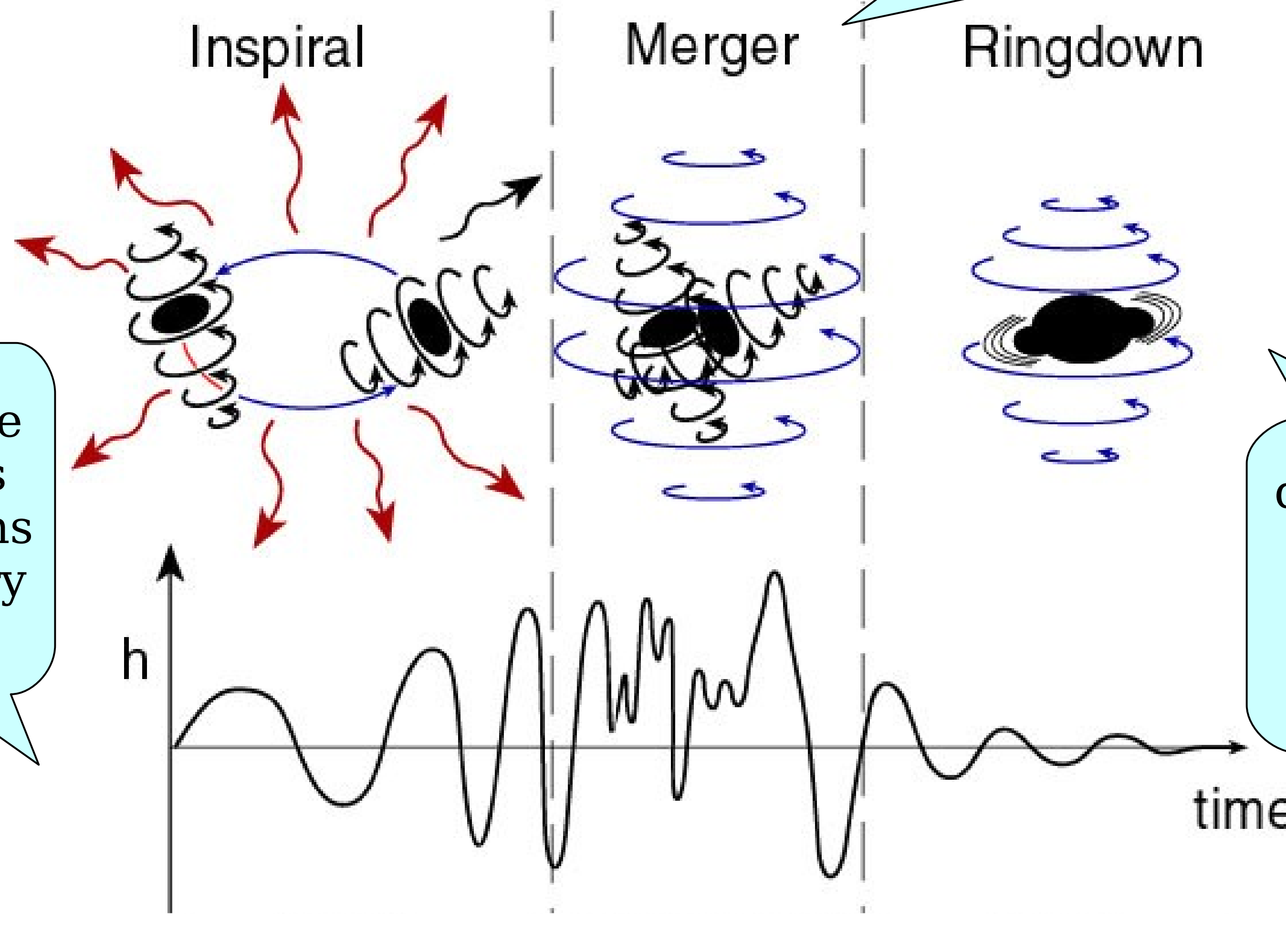




Evolution of a binary system

- GW produced in all three phases of this evolution.
- Waveforms and dynamics scale with BH masses and spins
  - — source modeling applicable to stellar BHs, IMBHs & MBHs....

strong-field spacetime dynamics, spin flips and couplings...

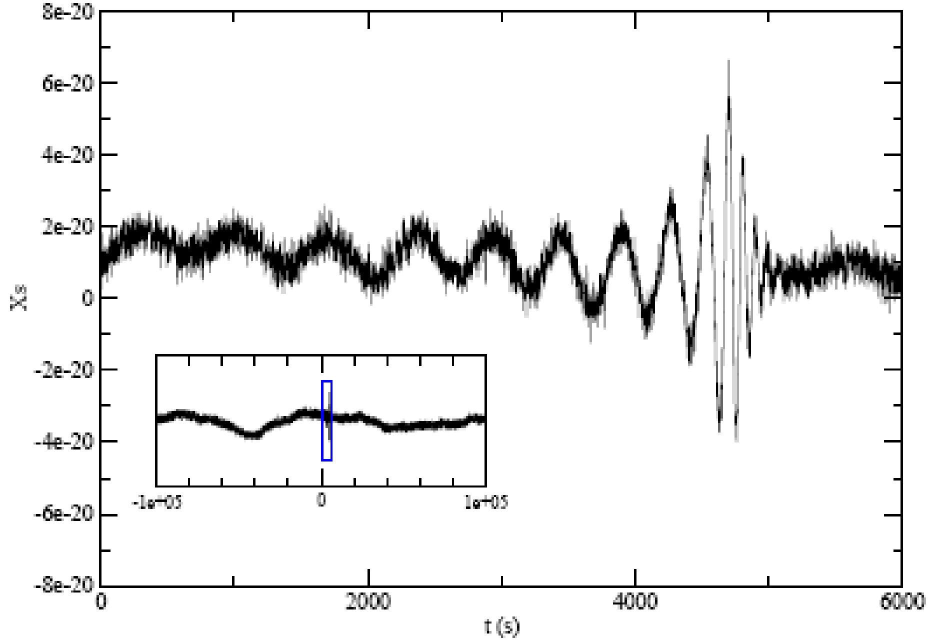


measure masses and spins of binary BHs

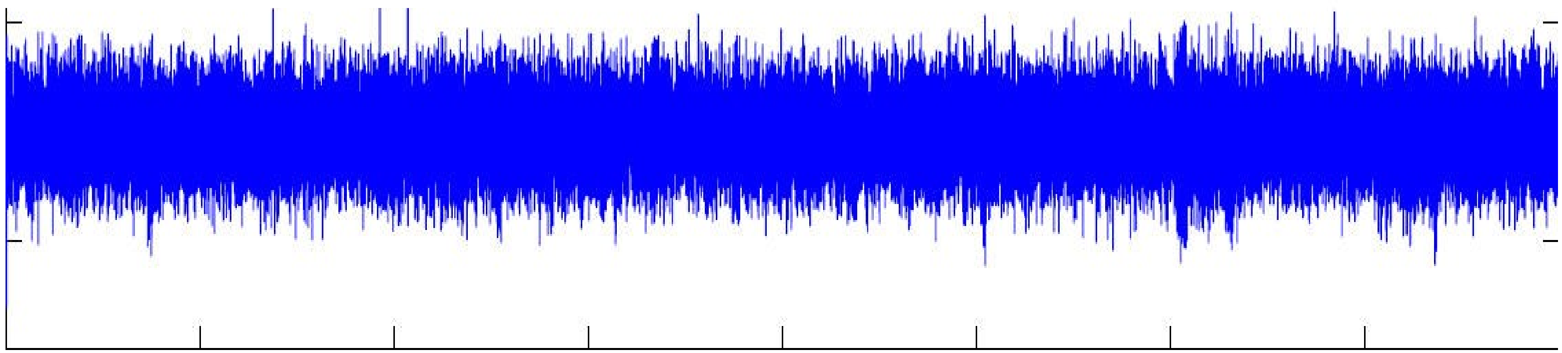
detect normal modes of ringdown to identify final Kerr BH

Why needs theoretical waveform?

- Expect



- Encounter



- Theoretical waveform (numerical relativity)

⇒ Data analysis: Matched Filtering ⇒ improve SNR

Initial Value & Time Evolution Challenges

THE EINSTEIN FIELD EQUATION

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Initial Value Problem:
The mathematical solution of the Hamiltonian & momentum constraints

Time Evolution
The mathematical solution of the Cauchy problem for the given initial data

Time-Independent Elliptic Equations

Time Dependent Hyperbolic Equations

Challenge
To find astrophysically realistic solutions

Challenge
To achieve stable evolution that conserves the constraints

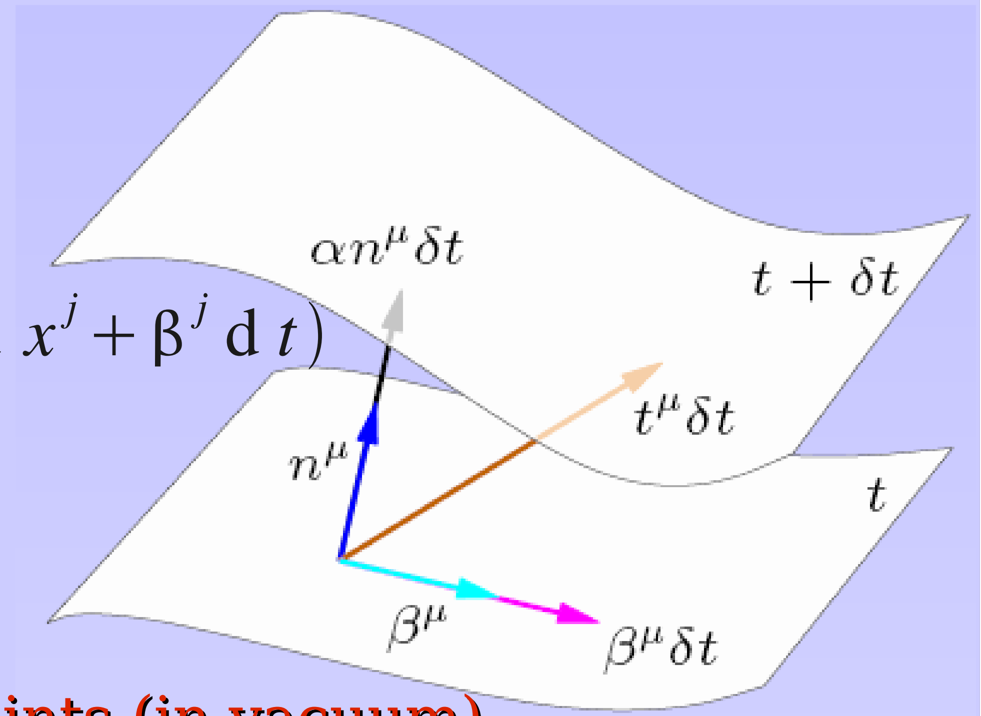
Necessary implementations for numerical relativity

- Einstein evolution equations solver (formulations)

  - slice 4D spacetime into a stack of 3D slices
  - typically solve 17 or more nonlinear, coupled PDEs

- Realistic initial conditions in GR
- Gauge conditions (coordinate conditions)
- Special techniques for handling BHs (eg. excision, puncture)
- Gravitational wave extraction techniques
- Powerful supercomputer & numerical techniques (eg.FMR/AMR)

3+1 Arnowitt-Deser-Misner (ADM) formulation



- $3+1 : {}^4M = R \times {}^3\Sigma$

- $ds^2 = -\alpha\, dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

  $\gamma_{ij}(t, x)$: metric

  $K_{ij}(t, x)$: extrinsic curvature

12 evolution equations & 4 constraints (in vacuum)

Constraints: $H \equiv R + K^2 - K_{ij} K^{ij} \simeq 0$

$$M_i \equiv \nabla_j K^j{}_i - \nabla_i K \simeq 0$$

Evolution: $\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij} \quad \Leftarrow \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} - \mathcal{L}_{\vec{\beta}}$

$$\frac{d}{dt} K_{ij} = \alpha\left(R_{ij} - 2K_{im}K^m{}_j + K K_{ij}\right) - \nabla_i \nabla_j \alpha$$

Baumgarte-Shapiro-Shibata-Nakamura (BSSN) Formulation

Features:

- 1^{st} derivative in time, 2^{nd} derivative in space
- A conformal decomposition of the metric and the traceless components of the extrinsic curvature.
- has been shown to be superior to the standard ADM formulation in terms of both accuracy and stability.
- Strongly hyperbolic with suitable gauges

BSSN variables:

$$
\begin{aligned}
\varphi &\equiv \frac{1}{12}\ln\gamma\,, & \tilde{\gamma}_{ij} &\equiv e^{-4\varphi}\gamma_{ij} \\
K &\equiv \gamma^{ij}K_{ij}\,, & \tilde{A}_{ij} &\equiv e^{-4\varphi}K_{\langle ij\rangle} \\
\tilde{\Gamma}^{i} &\equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^{i}_{jk}
\end{aligned}
$$

Field equations:

$$\frac{d}{dt}\varphi = -\frac{1}{6}\alpha K$$

$$\frac{d}{dt}\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}$$

$$\frac{d}{dt}K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \nabla^2\alpha$$

$$\frac{d}{dt}\tilde{A}_{ij} = \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) + e^{-4\varphi}\left(\alpha R_{\langle ij\rangle} - \nabla_{\langle i}\nabla_{j\rangle}\alpha\right)$$

$$\frac{\partial}{\partial t}\tilde{\Gamma}^i = 2\alpha\left(\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} + 6\tilde{A}^{ij}\varphi_{,j}\right) - 2\tilde{A}^{ij}\alpha_{,j}$$
$$+\tilde{\gamma}^{jk}\beta^i{}_{,jk} + \frac{1}{3}\tilde{\gamma}^{ij}\beta^k{}_{,jk} + \beta^j\tilde{\Gamma}^i{}_{,j} - \tilde{\Gamma}^j\beta^i{}_{,j} + \frac{2}{3}\tilde{\Gamma}^i\beta^j{}_{,j}$$

- The constraints of H & M^i were used to eliminate $\tilde{R}$ in $\frac{dK}{dt}$ and $\partial_j\tilde{A}^{ij}$ in $\partial_t\tilde{\Gamma}^i$.

Constraints

- Hamiltonian constraint:

$$H = e^{-4\varphi}\left(\tilde{R} - 8\,\tilde{\nabla}^2\varphi - 8\,\tilde{\nabla}^i\varphi\,\tilde{\nabla}_i\varphi\right) + \frac{2}{3}K^2 - \tilde{A}_{ij}\tilde{A}^{ij} \simeq 0$$

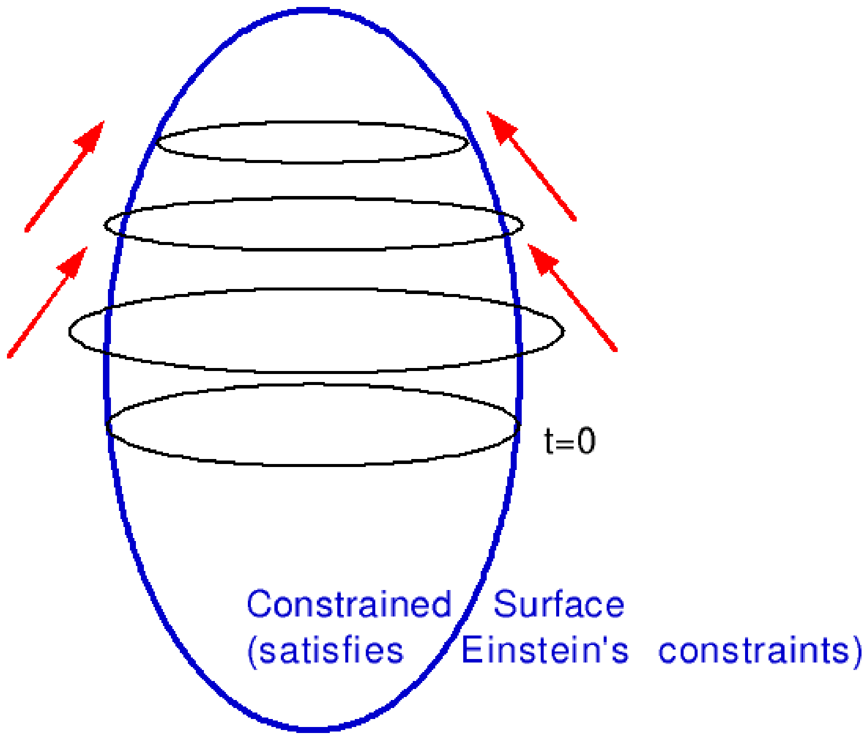
- Momentum constraints: $M^i = \tilde{\nabla}_j\tilde{A}^{ij} + 6\,\tilde{A}^{ij}\varphi_{,j} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} \simeq 0$

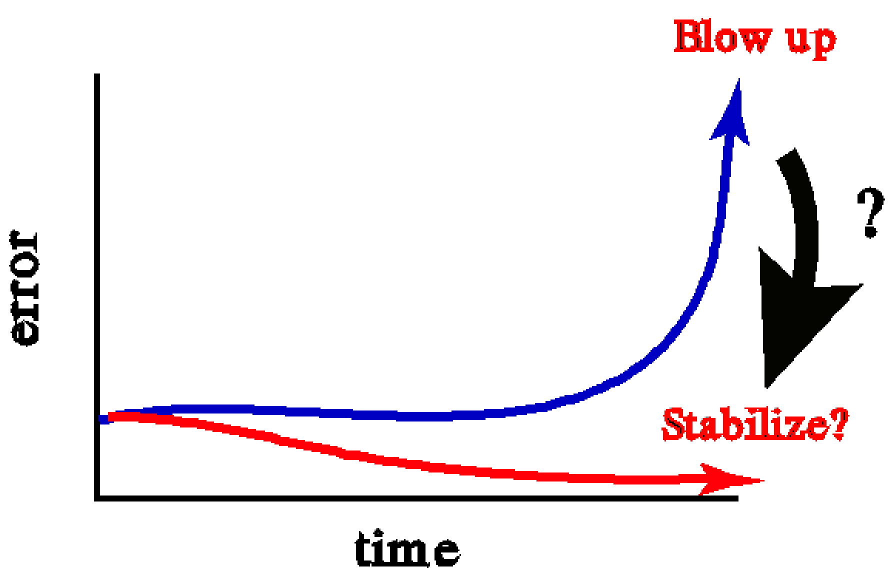
- Traceless constraint: $A \equiv \tilde{\gamma}^{ij}\tilde{A}_{ij} \simeq 0$

- Unimodular determinant constraint: $\tilde{\gamma} \equiv \det\left(\tilde{\gamma}_{ij}\right) \simeq 1$

- Gamma constraint: $G^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk} \simeq 0$

"Adjusted system" against a violation of constraints





- Control the violation of constraints by adjusting constraints to evolution equations.
- Eigenvalue analysis of constraint propagation equations may predict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic, ie, ADM & BSSN.

Remarks on stability issue from numerical experiments

For the adjusted systems:

- **Propagation:** $\partial_t u + \partial_x u = \cdots$
- **Linear terms:** $\partial_t u + f\, u = \cdots$ **(unadjusted)**

  $\partial_t u + f\, u + C(u) = \cdots$ **(adjusted)**

  $\Rightarrow \quad \partial_t u + f' u = \cdots \;\; \& \;\; f' > 0$

- **2^nd-order dissipation terms:** $\partial_t u \Leftarrow \partial_t u + C\left(\partial_i^2 u\right)$
- **higher-order dissipation terms:** $\partial_t u \Leftarrow \partial_t u + C\left(\partial_i^4 u\right)$

  (Kreiss-Oliger)

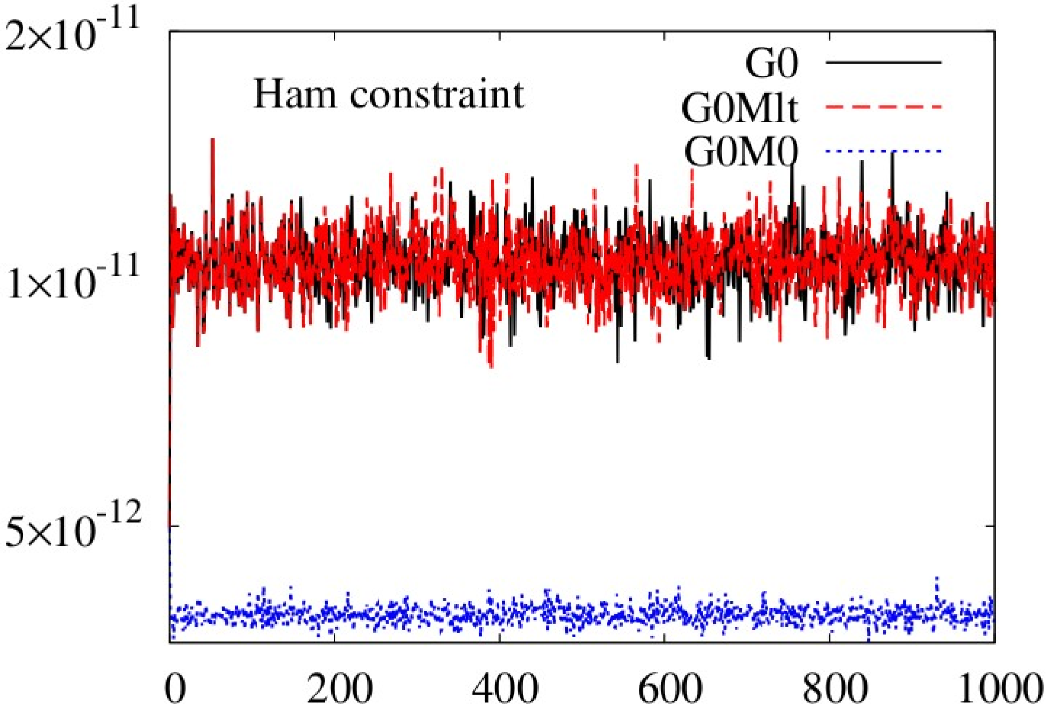
Modification 0

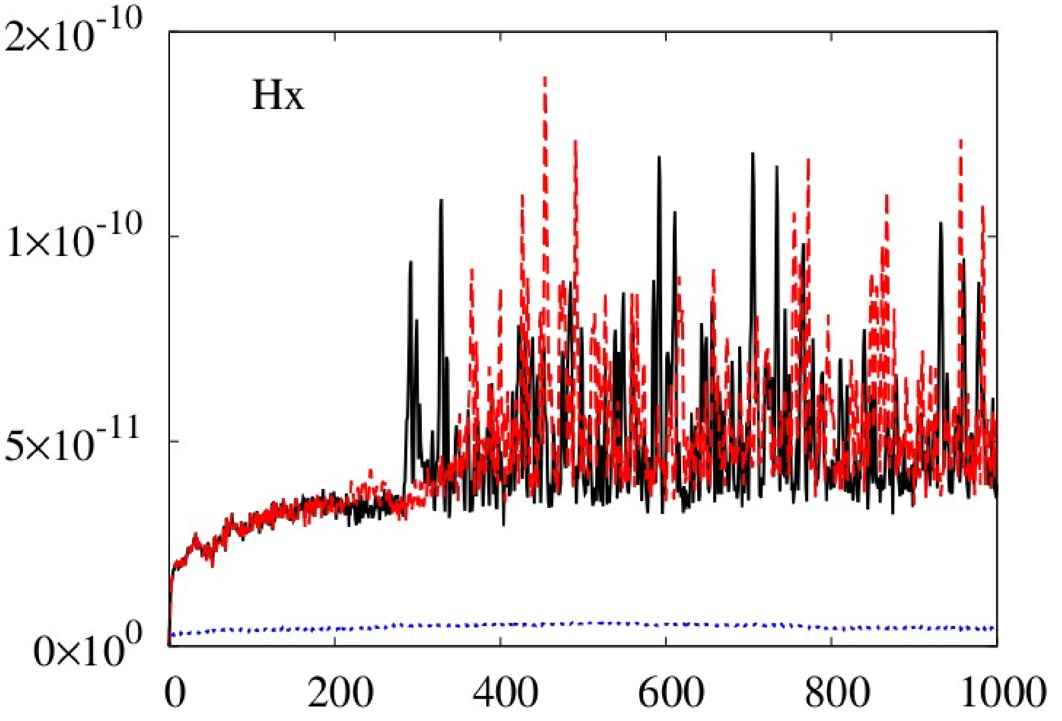
$$\tilde{\gamma} \simeq 1 \quad \Rightarrow \quad \tilde{\gamma}_{zz} = \frac{1 + \tilde{\gamma}_{yy}\,\tilde{\gamma}_{xz}^2 - 2\,\tilde{\gamma}_{xy}\,\tilde{\gamma}_{yz}\,\tilde{\gamma}_{xz} + \tilde{\gamma}_{xx}\,\tilde{\gamma}_{yz}^2}{\tilde{\gamma}_{xx}\,\tilde{\gamma}_{yy} - \tilde{\gamma}_{xy}^2}$$

$$A \simeq 0 \quad \Rightarrow \quad \tilde{A}_{yy} = -\frac{\tilde{A}_x{}^x + \tilde{A}_z{}^z + \tilde{A}_{xy}\,\tilde{\gamma}^{xy} + \tilde{A}_{yz}\,\tilde{\gamma}^{yz}}{\tilde{\gamma}^{yy}}$$

Advantage: correct one variable instead of six variables.

Disadvantage: the denominator could vanish.





Irreducible Decomposition

$$\tilde{\Gamma}^{i}{}_{jk} = \tilde{F}^{i}{}_{jk} + \frac{3}{5}\delta^{i}_{\langle j}\tilde{\Gamma}^{\ell}{}_{k\rangle\ell} - \frac{1}{5}\delta^{i}{}_{\langle j}\tilde{\Gamma}_{\mathbf{g}\,k\rangle} + \frac{1}{3}\tilde{\gamma}_{jk}\tilde{\Gamma}^{i}_{\mathbf{g}}$$

$$\tilde{\Gamma}^{i}_{\mathbf{g}} \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^{i}{}_{jk}, \qquad \tilde{\Gamma}^{\ell}{}_{i\ell} = \partial_i \ln\sqrt{\tilde{\gamma}} \simeq 0$$

$$\tilde{F}^{i}{}_{jk} = \text{the traceless part of } \quad \tilde{\Gamma}^{i}{}_{jk} = S^{i}{}_{jk} + A^{i}{}_{jk}$$

$$S_{ijk} = \tilde{F}_{(ijk)} = \frac{1}{3}\left(\tilde{F}_{ijk} + \tilde{F}_{jki} + \tilde{F}_{kij}\right)$$

$$A_{ijk} = \tilde{F}_{ijk} - S_{ijk} = \frac{2}{3}\left(\tilde{F}_{ijk} - \tilde{F}_{(jk)i}\right)$$

Constraint Application

$$\tilde{\Gamma}^i{}_{jk} \rightarrow \tilde{\Gamma}^i{}_{jk} + \frac{1}{3}\tilde{\gamma}_{jk} G^i - \frac{1}{5}\delta^i{}_{\langle j} G_{k\rangle} - \frac{3}{5}\delta^i_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell}$$

Similarly,

$$\partial_i \tilde{\gamma}_{jk} \rightarrow \partial_i \tilde{\gamma}_{jk} - \frac{1}{3}\tilde{\gamma}_{jk}\tilde{\Gamma}^\ell{}_{i\ell} + \frac{1}{5}\tilde{\gamma}_{i\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell} + \frac{3}{5}\tilde{\gamma}_{i\langle j} G_{k\rangle}$$

$$\Rightarrow \quad \beta^i \partial_i \tilde{\gamma}_{jk} \rightarrow \beta^i \partial_i \tilde{\gamma}_{jk} + \frac{3}{5}\beta_{\langle j} G_{k\rangle} \quad \Leftarrow \quad \tilde{\Gamma}^\ell{}_{i\ell} \text{ parts have negligible effects}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}_{ij} \rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}_{ij} + \frac{3}{5}\beta_{\langle j} G_{k\rangle}$$

Modification I

$$\tilde{\Gamma}^i{}_{jk} \rightarrow \tilde{\Gamma}^i{}_{jk} + \frac{1}{3}\tilde{\gamma}_{jk} G^i - \frac{1}{5}\delta^i{}_{\langle j} G_{k\rangle} - \frac{3}{5}\delta^i_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell} \Rightarrow \tilde{\Gamma}^i{}_{jk}$$

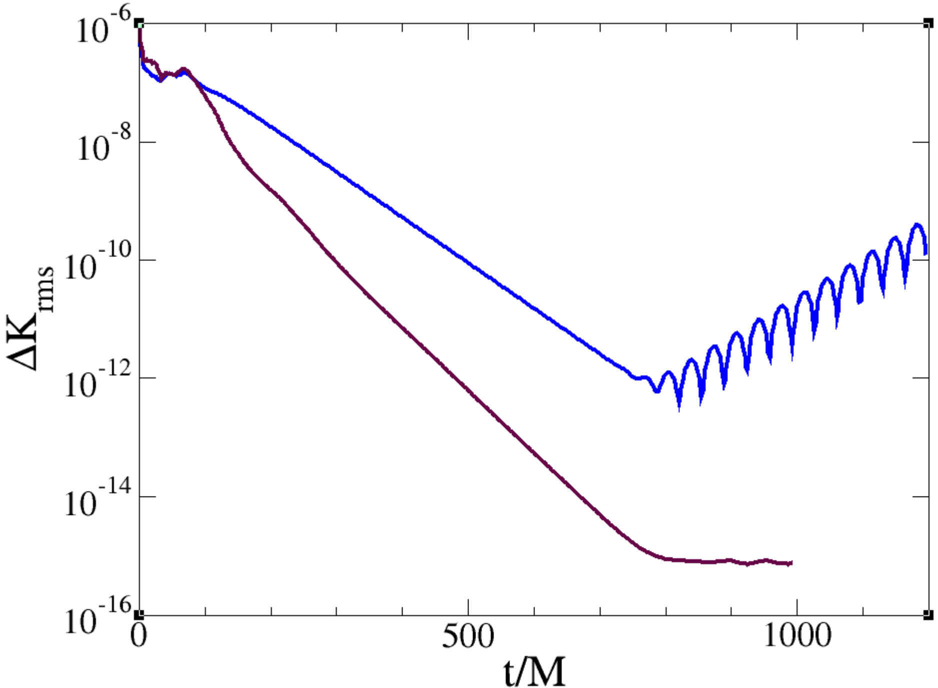
$$\frac{\partial}{\partial t}\tilde{\Gamma}^i = 2\alpha\left(\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} - \frac{2}{3}\left(\tilde{\gamma}^{ij}K\right)_{,j} + 6\tilde{A}^{ij}\varphi_{,j}\right) - 2\tilde{A}^{ij}\alpha_{,j} + \tilde{\partial}^2\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\beta^k{}_{,jk}$$
$$+ \beta^j\tilde{\Gamma}^i{}_{,j} - \tilde{\Gamma}^j\beta^i{}_{,j} + \frac{2}{3}\left(\beta^j{}_{,j} - 2\alpha K\right)\tilde{\Gamma}^i - (1+\xi)\,\Theta\left(\lambda^i\right)\lambda^i G^i$$

where $$\lambda^i = \frac{2}{3}\left(\beta^j{}_{,j} - 2\alpha K\right) - \beta^{\hat{i}}{}_{,\hat{i}} - \frac{2}{5}\alpha\tilde{A}_{\hat{i}}{}^{\hat{i}}$$

Advantage: stabilize the code without the need of the substitute

$$\tilde{\Gamma}^i \rightarrow \tilde{\Gamma}^i_{\mathbf{g}} \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk}$$

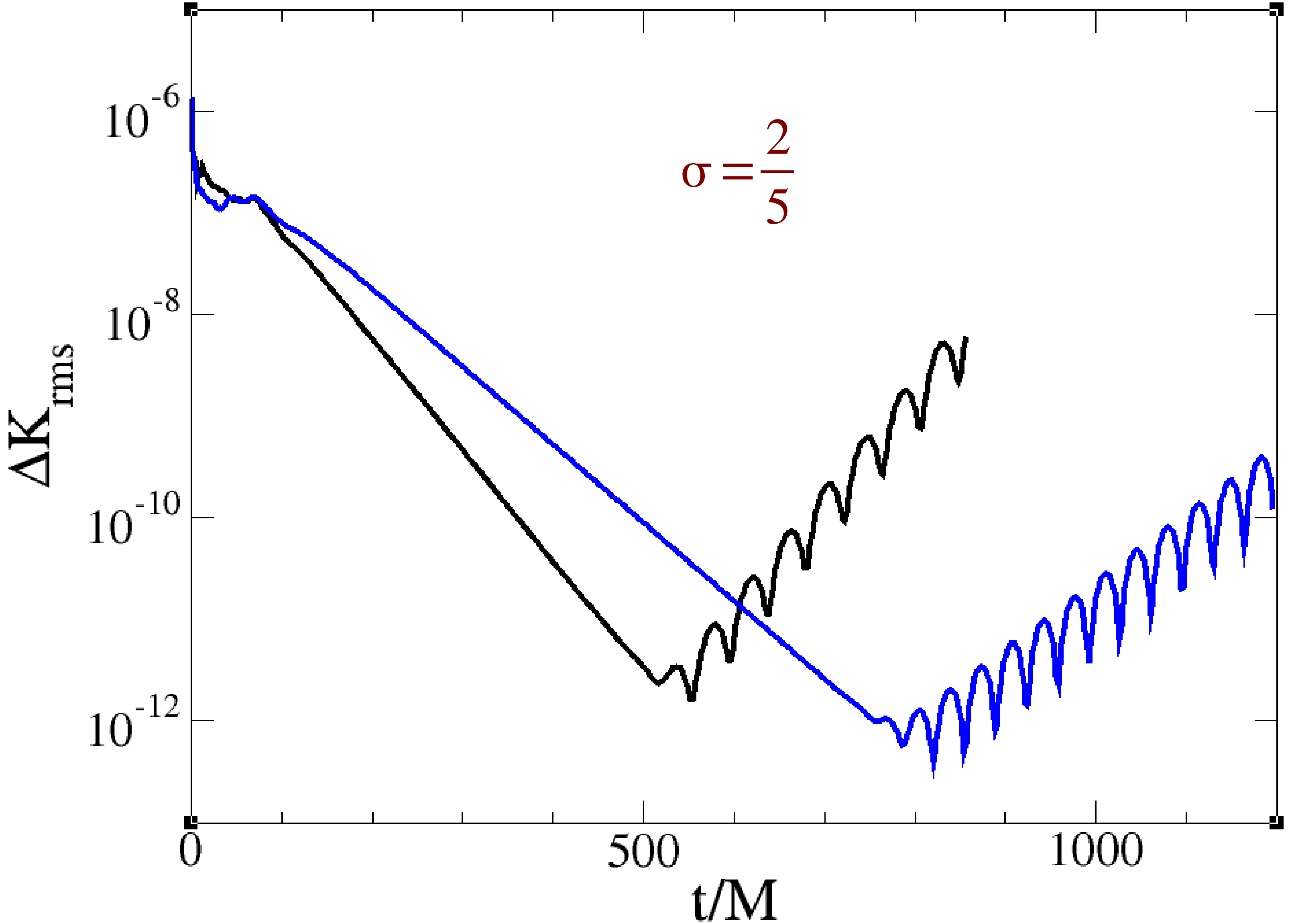
Disadvantage: one parameter to choose



Modification II

1st derivative of the conformal metric

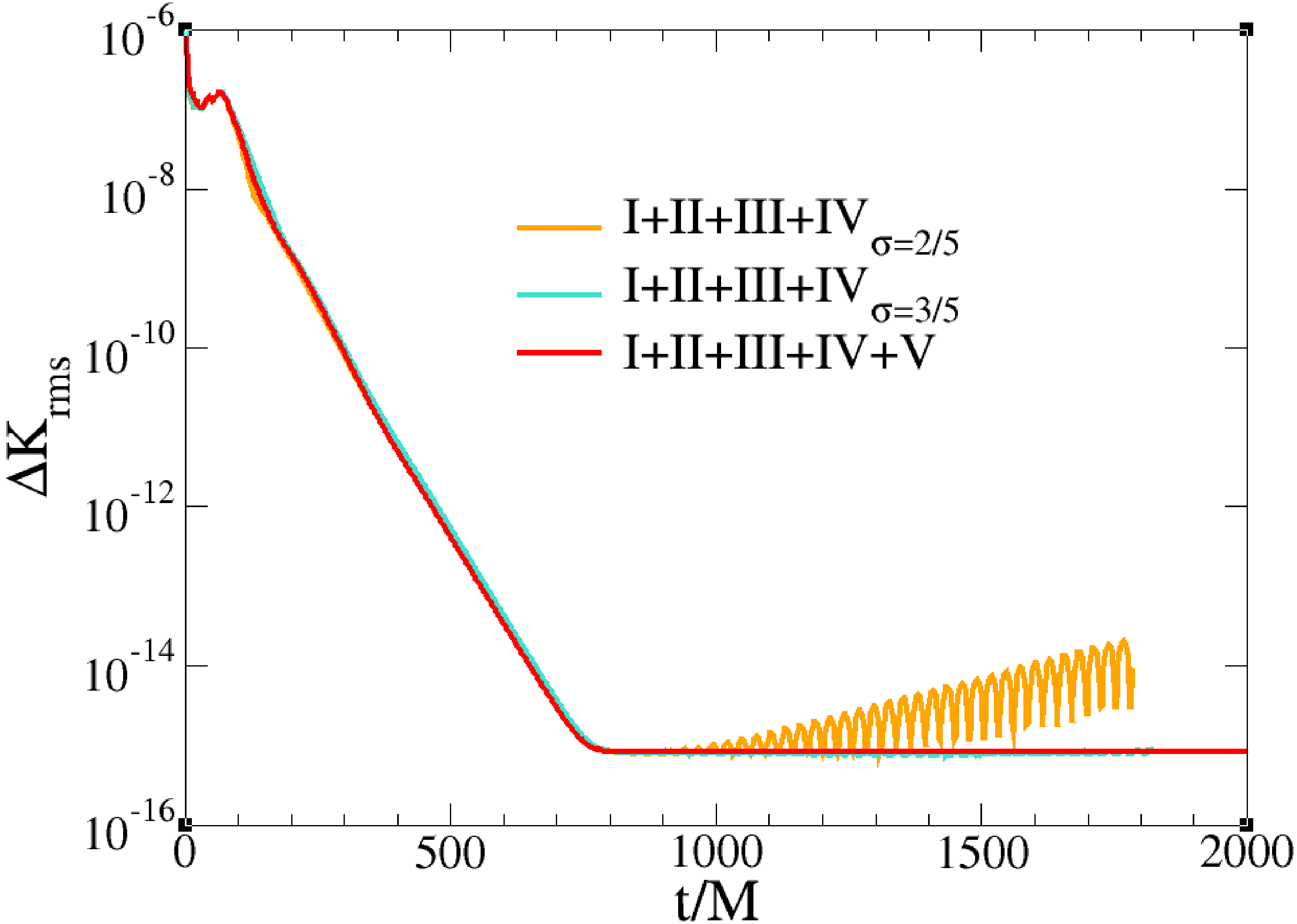
$$\frac{d}{dt}\tilde{\gamma}_{ij} \rightarrow \frac{d}{dt}\tilde{\gamma}_{ij} + \sigma\beta_{(j}G_{k)} - \frac{1}{5}\tilde{\gamma}_{jk}\beta^i G_i$$



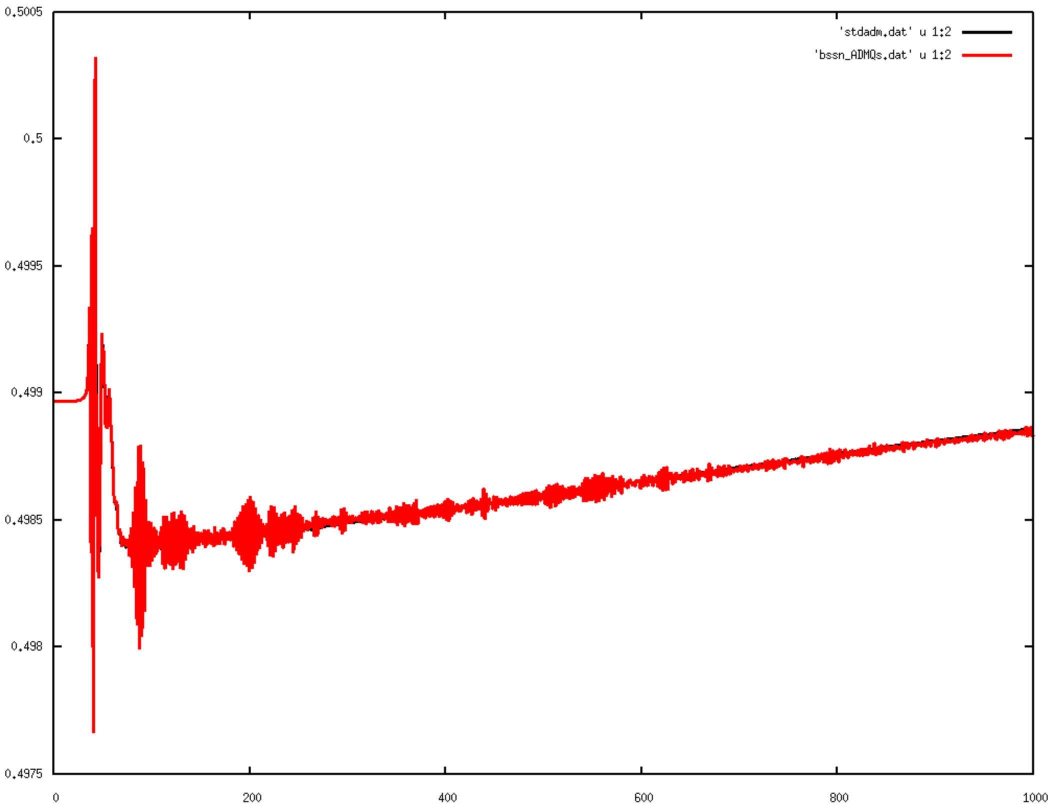
Modification III

- Yoneda & Shinkai ['02]:

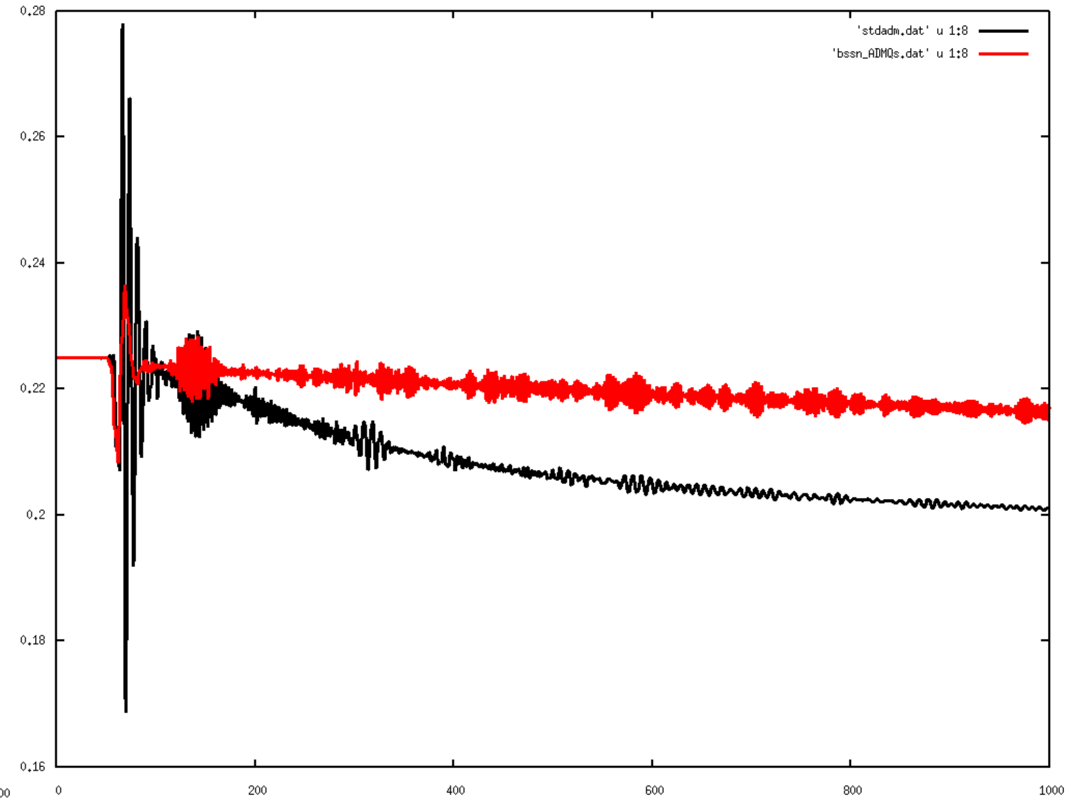
$$\frac{d}{dt}\tilde{A}_{ij} \rightarrow \frac{d}{dt}\tilde{A}_{ij} + \kappa_A \alpha \tilde{\nabla}_{(i} M_{j)} \quad \Rightarrow \quad \frac{d}{dt}\tilde{A}_{ij} \rightarrow \frac{d}{dt}\tilde{A}_{ij} + h f(\alpha) M_{\langle i,j \rangle}$$



Single black hole with *a*=0.9

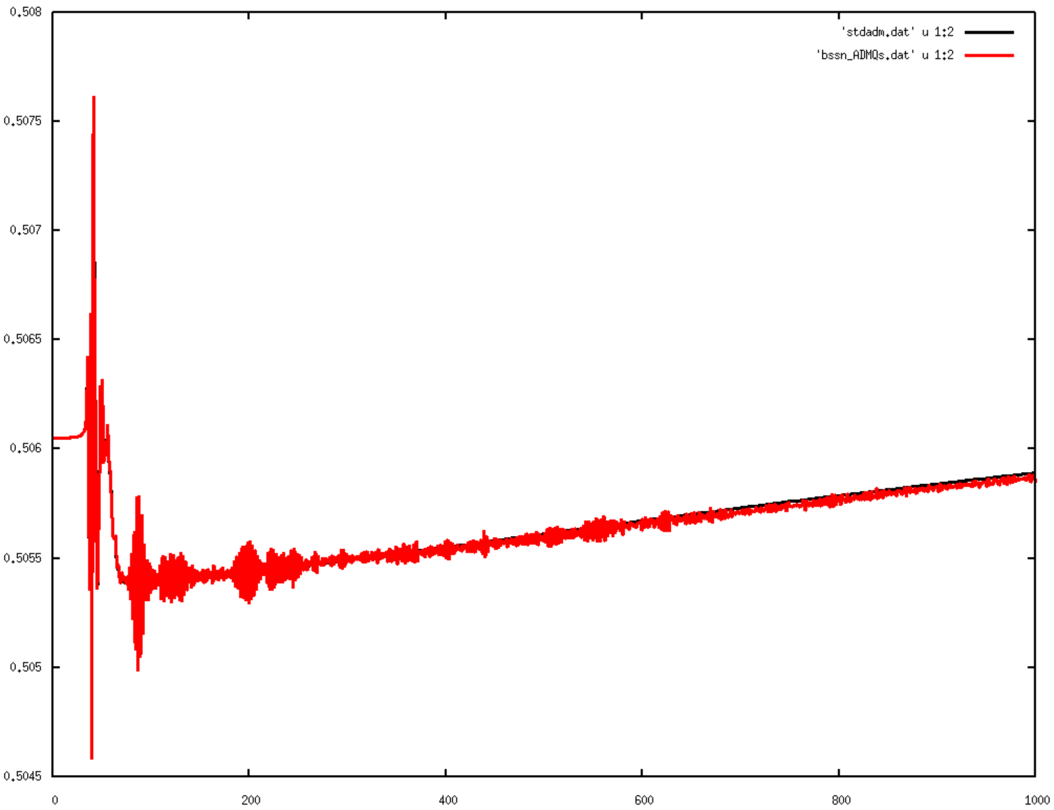


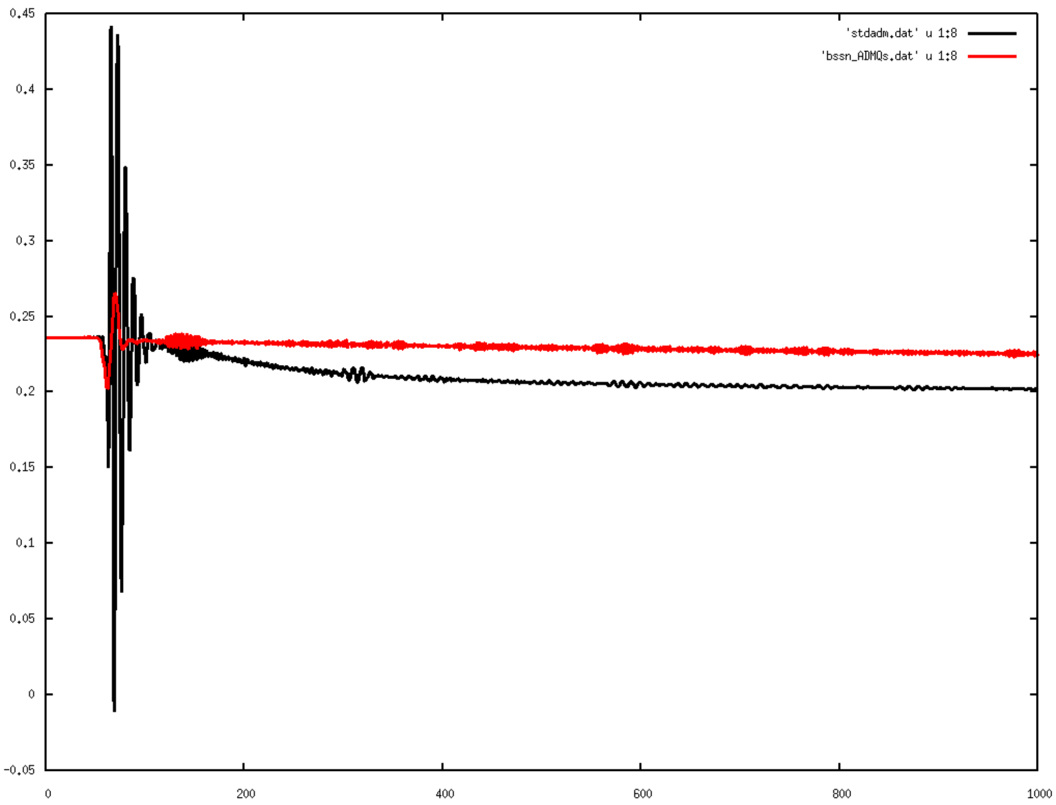
ADM mass



Angular momentum

Single black hole with *a*=0.9275

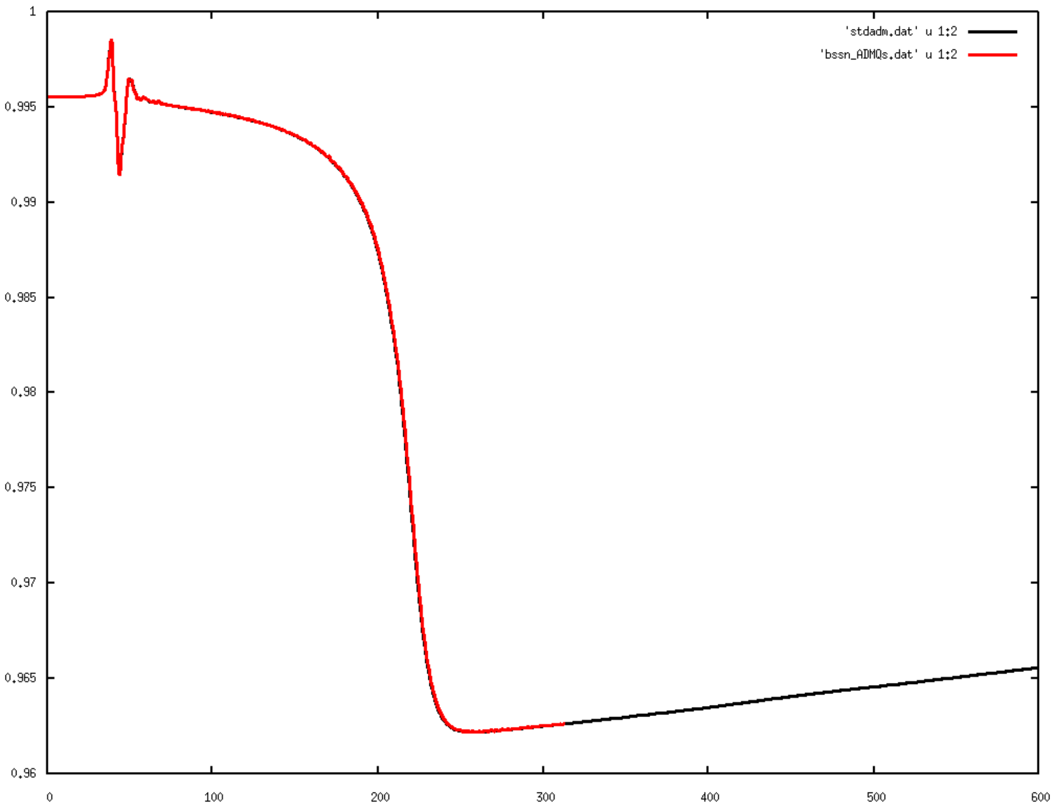




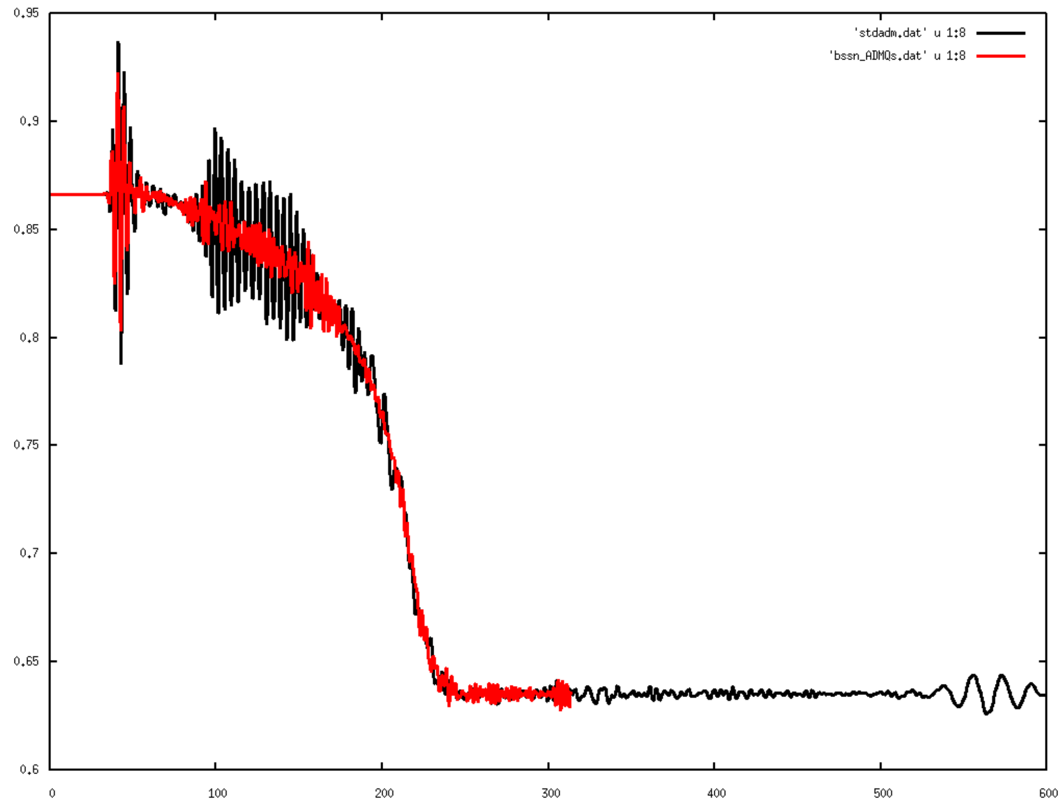
ADM mass

Angular momentum

Binary black hole with *a*=0

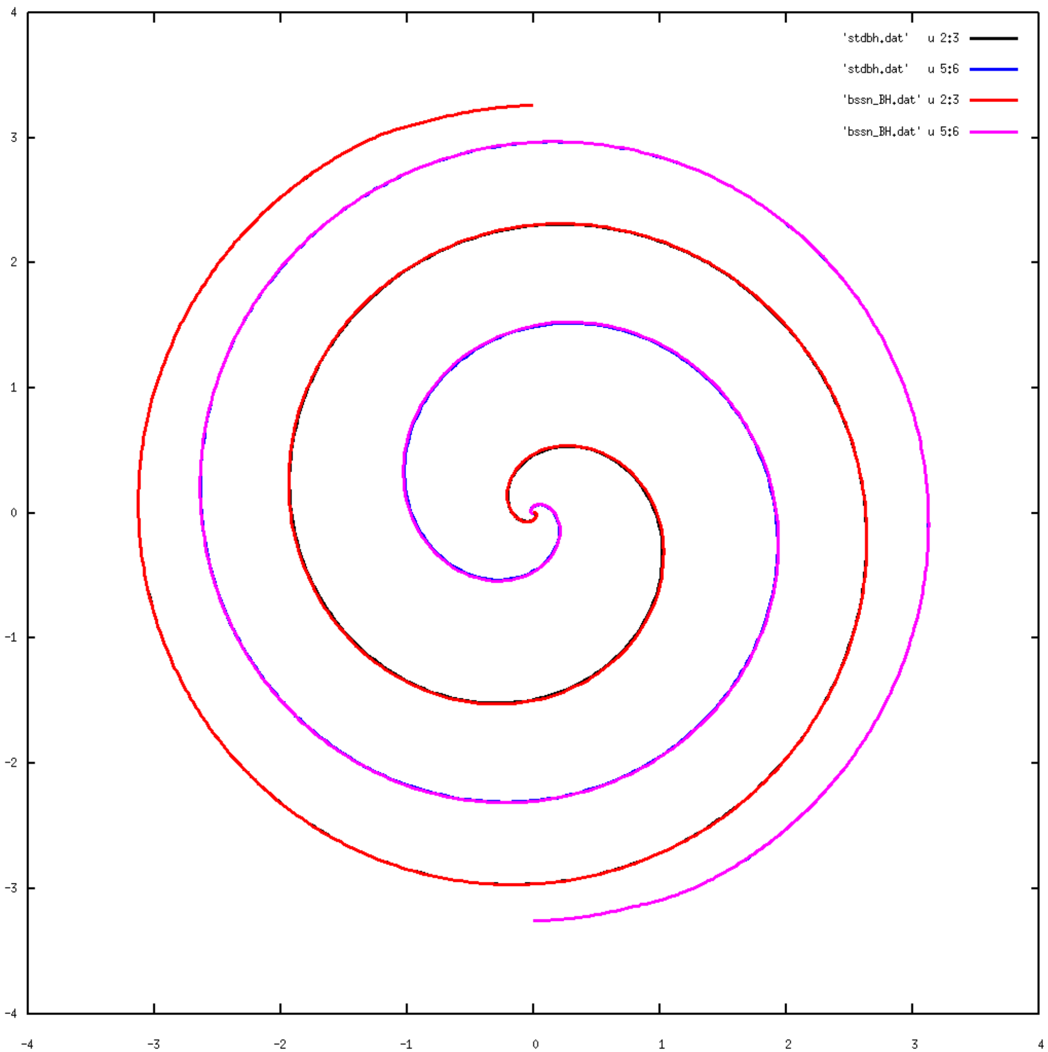


ADM mass

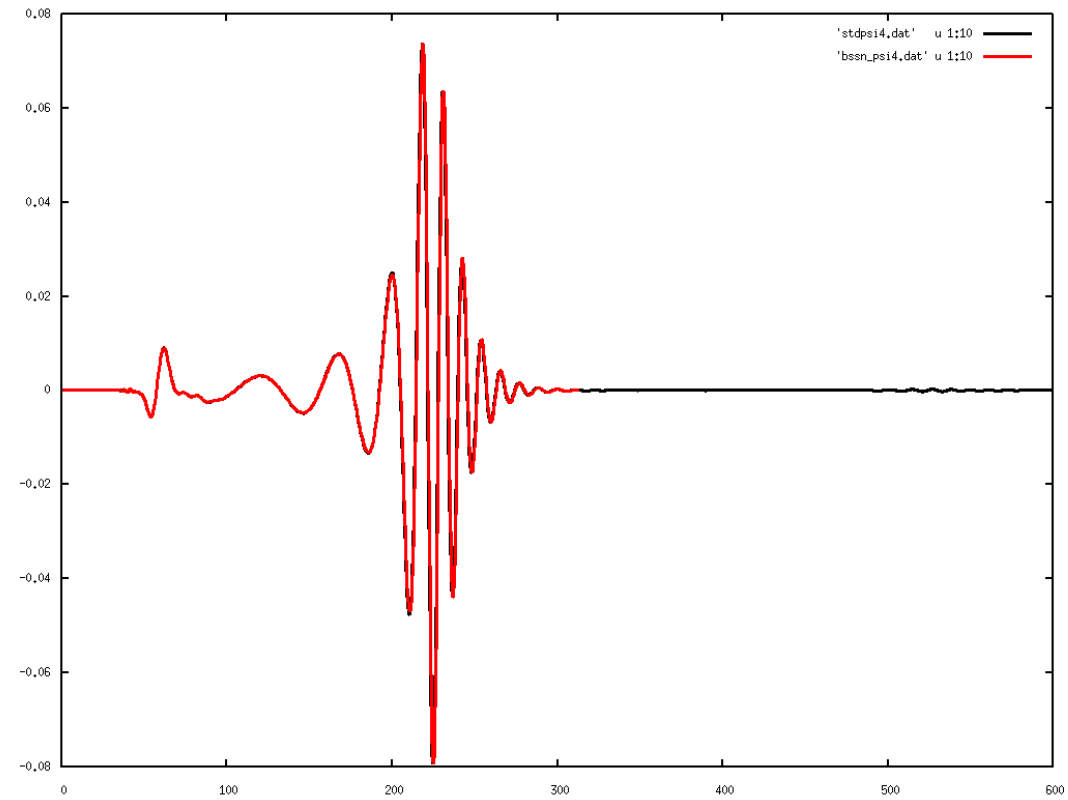


Angular momentum

Binary black hole with *a*=0

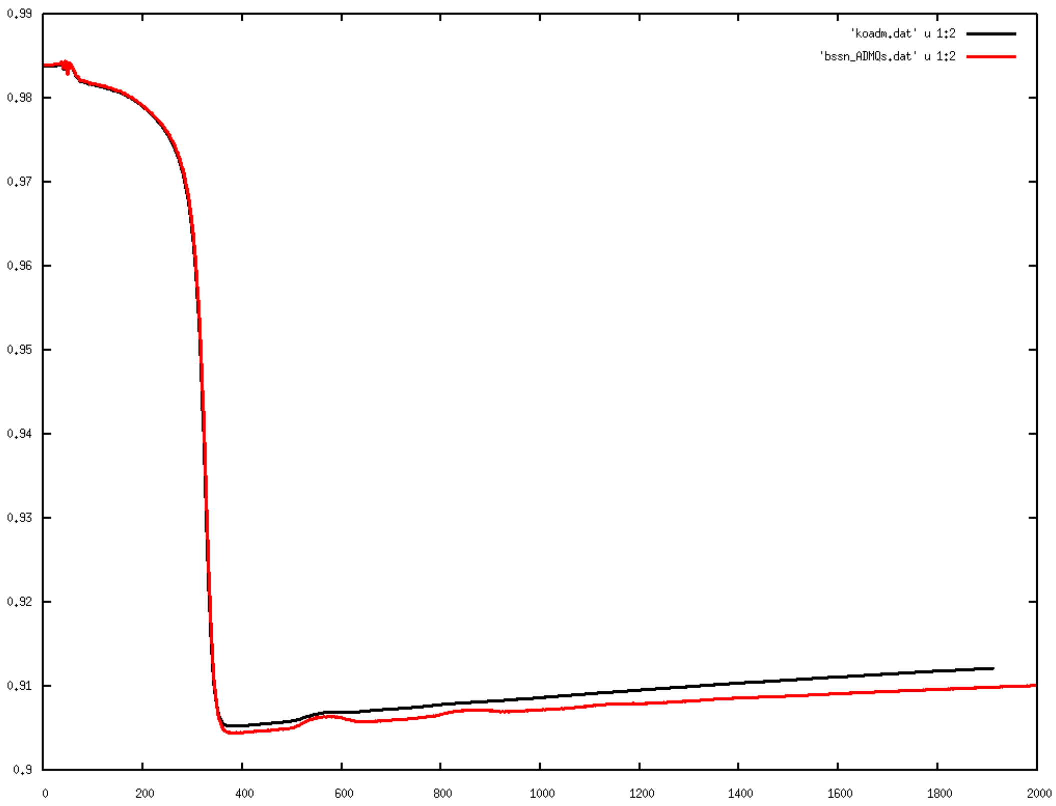


Black hole trajectory

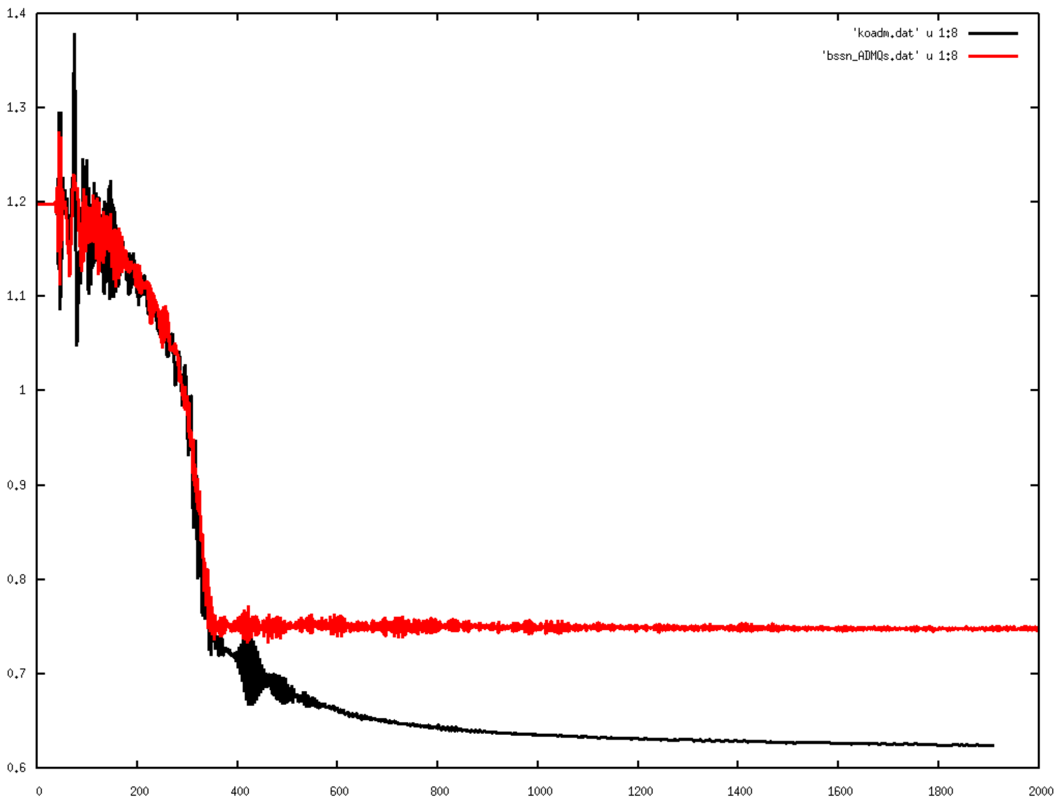


waveform

Binary black hole with a=0.9

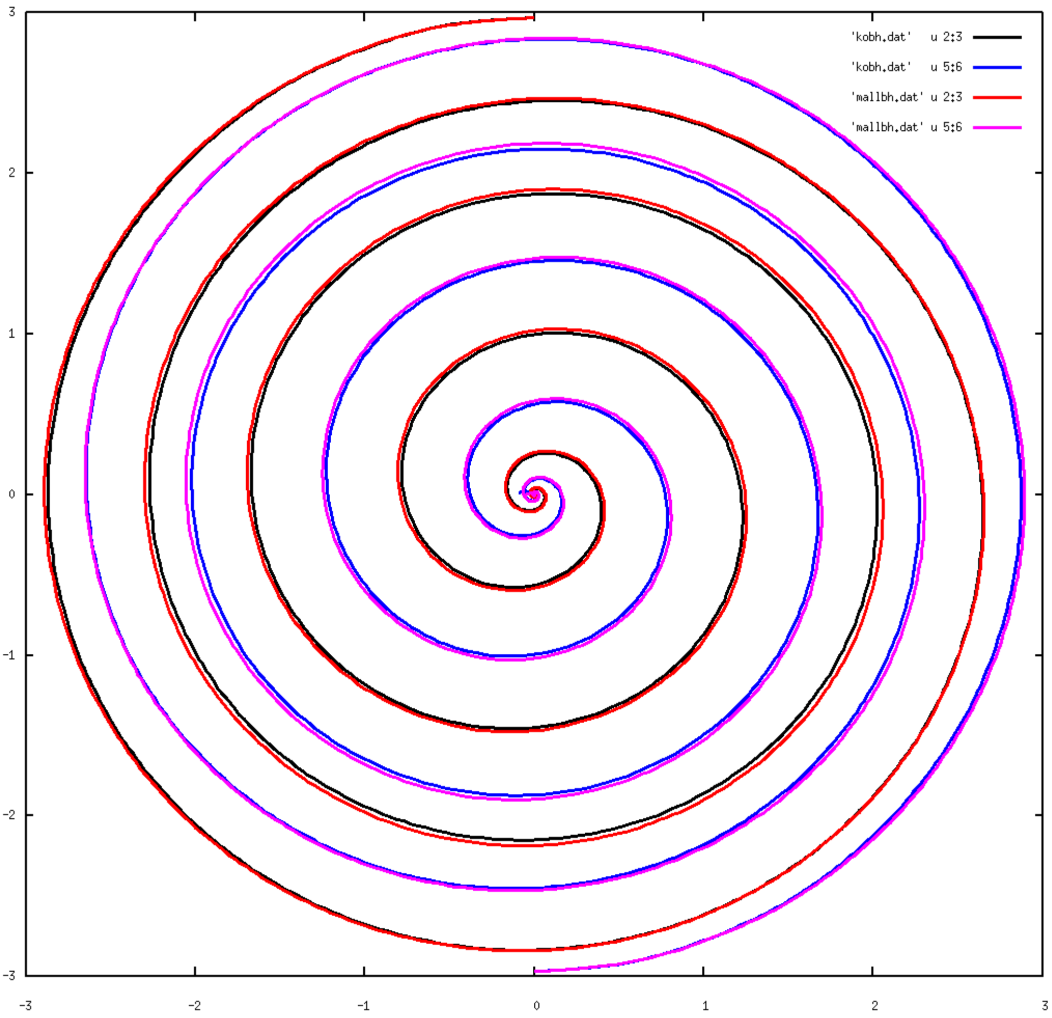


ADM mass

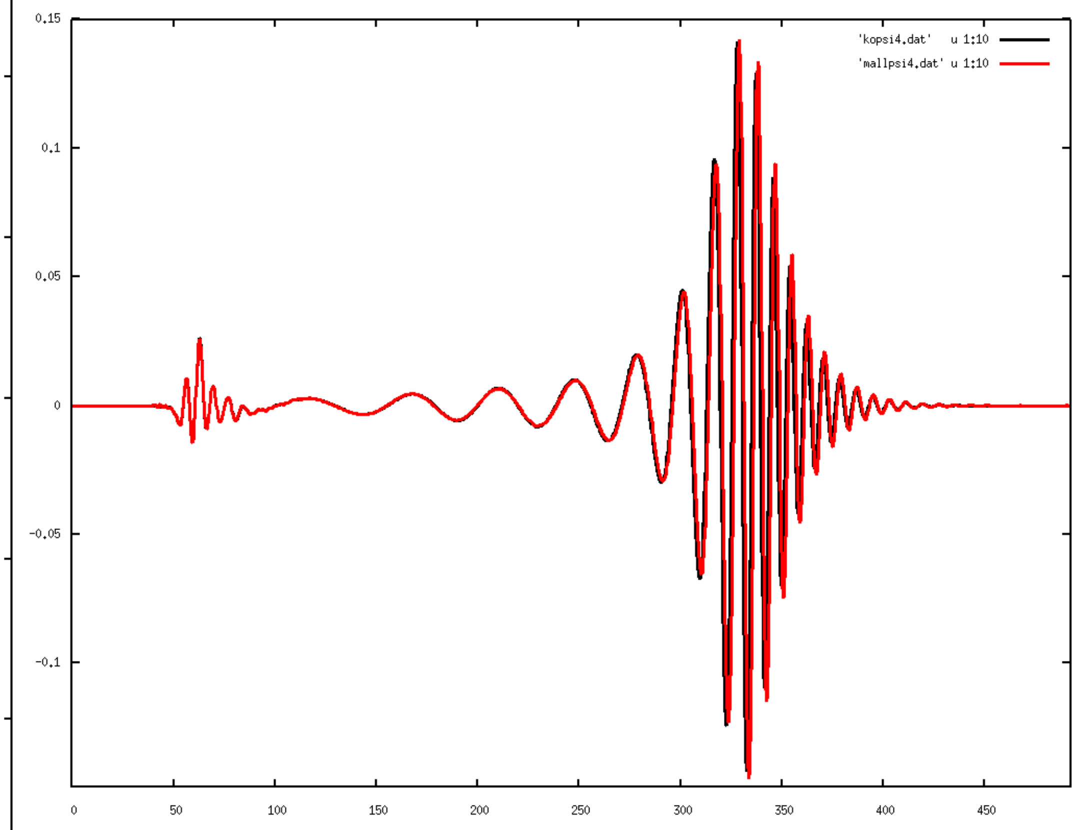


Angular momentum

Binary black hole with *a*=0.9



Black hole trajectory



waveform

Summary

- The modifications focus on the BSSN physical variables

- The recipes are able to suppress instability efficiently and thus increase accuracy in both single BH and binary BBH.

- These modifications can be applied in binary neutron star cases.

- Plan to test these recipes the extremely mass ratio inspiral (EMRI).